

# Analysis of Multiple Time Series

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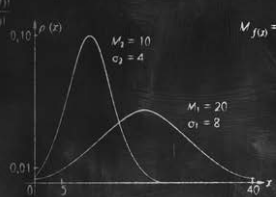
This version: February 25, 2019

February 21, 2019



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

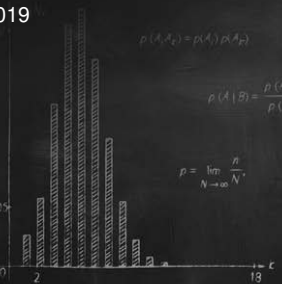


$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

$$S = v_f^2 + \frac{\sigma^2}{2}$$

$$F = G \frac{m_1 m_2}{\rho^2}$$

$$f(v) = 4\pi \left( \frac{m_0}{2\pi kT} \right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}$$



$$\sigma_n = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}}$$

$$D_x = \sigma^2 = M_{x^2} - (M_x)^2$$

$$p_T(\lambda) = \frac{\lambda^c}{c!} e^{-\lambda}$$

$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^c p_i x_i$$

$$D_x = \sum_{i=1}^c p_i (x_i - M_x)^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = 4\sqrt{\frac{k^3}{\pi}} v^2 e^{-kv^2}$$

$$\phi(\ln x) d(\ln x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} d(\ln x) = \frac{1}{\sqrt{2\pi\sigma x}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx$$

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

$$p = \lim_{N \rightarrow \infty} \frac{n}{N}$$

$$\langle v \rangle = \frac{\langle v \rangle t}{n\sqrt{2\pi}d^2}$$

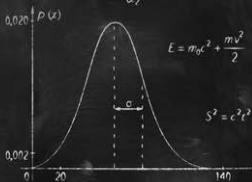
$$C = \frac{\epsilon\epsilon_0 S}{d}$$



$$B = \frac{U_0 I}{2\pi b} (\cos \alpha_1 - \cos \alpha_2)$$

$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\Phi_2 - \Phi_1)$$

$$h\nu = A + \frac{mv^2}{2}$$



$$E = m_0 c^2 + \frac{mv^2}{2}$$

$$m = m_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$S^2 = c^2 t^2 - l^2 = \ln v$$

$$r_n = \frac{4\pi\epsilon_0 n^2 a^2}{mZe^2}$$

# This week's material



$$D_t = \int_{-\infty}^{\infty} (x - M_t) f(x) dx$$

- Vector Autoregressions
- Basic examples
- Properties
  - ▶ Stationarity
- Revisiting univariate ARMA processes
- Forecasting
  - ▶ Granger Causality
  - ▶ Impulse Response functions
- Cointegration
  - ▶ Examining long-run relationships
  - ▶ Determining whether a VAR is cointegrated
  - ▶ Error Correction Models
  - ▶ Testing for Cointegration
    - Engle-Granger

Lots of revisiting univariate time series.

$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$$

$$P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

$$A_n^k = \frac{n!}{(n-k)!}$$

$$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

$$\lambda_n^k = \underbrace{n \cdot n \cdot \dots \cdot n}_k = n^k$$

$$\tilde{C}_n^m = P_{m, n-m-1} = \frac{(n+m-1)!}{m!(n-1)!}$$

$$\tilde{P}_{n_1, n_2, \dots, n_k} = \frac{(n_1 + n_2 + \dots + n_k)!}{n_1! n_2! \dots n_k!}$$

$$C_n^k = \frac{n!}{k!(n-k)!}$$

$$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + p(B|A_3)p(A_3) + \dots + p(B|A_n)p(A_n)$$

$$p(x) = \frac{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}$$



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

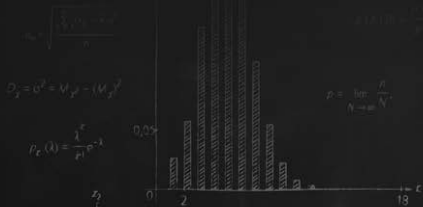
$$S = v_0 t + \frac{at^2}{2}$$

$$E = G \frac{m_1 m_2}{Q^2}$$

$$f(v) = A \exp\left(-\frac{mv}{kT}\right) \exp\left(-\frac{mv^2}{2kT}\right)$$



# Impulse Response Functions



$$D_x = \sigma^2 = M_x^2 - (M_x)^2$$

$$p_x(x) = \frac{1}{\sigma} e^{-x/\sigma}$$

$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^n p_i x_i$$

$$D_x = \sum_{i=1}^n p_i (x_i - M_x)^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = A \sqrt{\frac{k^3}{\pi}} \sqrt{v} e^{-kv^2}$$

$$v = \frac{2V}{n \sqrt{2\pi} d^2}$$

$$C = A \cos \alpha \frac{2V}{n \sqrt{2\pi} d^2}$$

$$C = \frac{v \pi S}{d}$$



$$d = \frac{R \sin \alpha}{\sin \beta}$$

$$d^2 = d_1^2 + d_2^2 + 2 d_1 d_2 \cos(\beta_1 - \beta_2)$$

$$nV = A + \frac{mV^2}{2}$$

$$0.020 \rho(v)$$



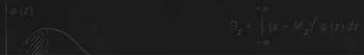
$$E = m_0 c^2 + \frac{m^2 v^2}{2}$$

$$m = m_0 \sqrt{1 - \beta^2}$$

$$S^2 = c^2 t^2 - l^2 = in v$$

$$r_n = \frac{4\pi \epsilon_0 n^2 n^2}{m^2 c^2}$$

# Impulse Response Functions



- Second fundamentally new concept
- Complicated dynamics of a VAR make direct interpretation of coefficients difficult
- Solution is to examine impulse responses
- The impulse response function of  $y_i$  with respect to a shock in  $\epsilon_j$ , for any  $j$  and  $i$ , is defined as the change in  $y_{it+s}$ ,  $s \geq 0$  for a unit shock in  $\epsilon_{jt}$ 
  - ▶ Hard to decipher
- As long as  $\mathbf{y}_t$  is covariance stationarity it must have a VMA representation,

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\epsilon}_t + \boldsymbol{\Xi}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\Xi}_2 \boldsymbol{\epsilon}_{t-2} + \dots$$

- $\boldsymbol{\Xi}_j$  are the impulse responses!
- Why?
  - ▶ Directly measure the effect in period  $j$  of any shock



$$\theta_i = \int_{-\pi}^{\pi} (1 - M_2 / \alpha) dx$$

- Any stationary AR(P)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_P y_{t-P} + \epsilon_t$$

can be represented as an MA( $\infty$ )

$$y_t = \phi_0 / (1 - \phi_1 - \phi_2 - \dots - \phi_P) + \epsilon_t + \sum_{i=1}^{\infty} \theta_i \epsilon_{t-i}$$

- AR(1)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$

becomes

$$y_t = \phi_0 / (1 - \phi_1) + \epsilon_t + \sum_{i=1}^{\infty} \phi_1^i \epsilon_{t-i}$$

- Stationary VAR(P) have the same relationship to VMA( $\infty$ )

$$\mathbf{y}_t = \Phi_0 + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \dots + \Phi_P \mathbf{y}_{t-P} + \epsilon_t$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \epsilon_t + \Xi_1 \epsilon_{t-1} + \Xi_2 \epsilon_{t-2} + \dots$$

$$R^2 = \frac{n!}{(n-k)!}$$



$$Q_j = \int_{-\infty}^{\infty} (x - \mu_j)^j \phi(x) dx$$

- Easy in VAR(1)

$$\mathbf{y}_t = (\mathbf{I}_K - \Phi_1)^{-1} \Phi_0 + \epsilon_t + \Phi_1 \epsilon_{t-1} + \Phi_1^2 \epsilon_{t-2} + \dots$$

- $\Xi_j = \Phi_1^j$
- In the general VAR(P),

$$\Xi_j = \Phi_1 \Xi_{j-1} + \Phi_2 \Xi_{j-2} + \dots + \Phi_P \Xi_{j-P}$$

where  $\Xi_0 = \mathbf{I}_k$  and  $\Xi_m = \mathbf{0}$  for  $m < 0$ .

- In a VAR(2),

$$\mathbf{y}_t = \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \epsilon_t$$

$$- \Xi_0 = \mathbf{I}_k, \Xi_1 = \Phi_1, \Xi_2 = \Phi_1^2 + \Phi_2, \text{ and } \Xi_3 = \Phi_1^3 + \Phi_1 \Phi_2 + \Phi_2 \Phi_1.$$

- Confidence intervals are also somewhat painful
  - Explained in notes

# Considerations for Shocks

- Simple bivariate VAR(1)

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- How you *shock* matters
- Depends on correlation between  $\epsilon_{1,t}$  and  $\epsilon_{2,t}$
- 3 methods
  - ▶ Ignore correlation and just shock  $\epsilon_{j,t}$  with a 1 standard deviation shock
  - ▶ Use Cholesky to factor  $\Sigma$  and use  $\Sigma^{1/2} \mathbf{e}_j$  where  $\mathbf{e}_j$  is a vector of zeros with 1 in the  $j^{\text{th}}$  position

$$\Sigma = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix} \quad \Sigma_C^{1/2} = \begin{bmatrix} 1 & 0 \\ .5 & .866 \end{bmatrix}$$

- Variable order matters
- ▶ “Generalized” impulse response that uses a projection method

# Example of the different shocks

- Define the error covariance

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_x \sigma_y \rho \\ \sigma_x \sigma_y \rho & \sigma_y^2 \end{bmatrix}$$

- Standardized

$$\begin{bmatrix} \sigma_x \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ \sigma_y \end{bmatrix}$$

- Cholesky

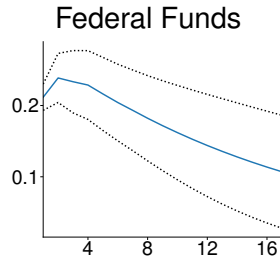
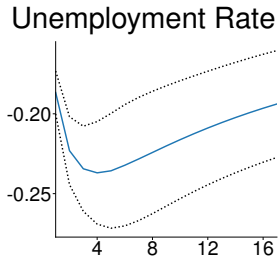
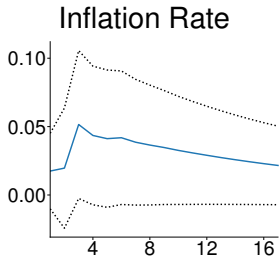
$$\Sigma_C^{1/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_x & 0 \\ \sigma_y \rho & \sigma_y \sqrt{1 - \rho^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma_y \sqrt{1 - \rho^2} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x & 0 \\ \sigma_y \rho & \sigma_y \sqrt{1 - \rho^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_x \\ \sigma_y \rho \end{bmatrix}, \text{ other is } \begin{bmatrix} 0 \\ \sigma_y \sqrt{1 - \rho^2} \end{bmatrix}$$



# Impulse Responses

- Federal Funds ordered first
- Response to Federal Funds Shock
- Cholesky factorization



$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$$

$$P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

$$A_n^k = \frac{n!}{(n-k)!}$$

$$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

$$\lambda_n^k = \underbrace{n \cdot n \cdot \dots \cdot n}_k = n^k$$

$$\tilde{C}_n^m = P_{m, n-m-1} = \frac{(n+m-1)!}{m!(n-1)!}$$

$$\tilde{P}_{n_1, n_2, \dots, n_k} = \frac{(n_1 + n_2 + \dots + n_k)!}{n_1! n_2! \dots n_k!}$$

$$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + p(B|A_3)p(A_3) + \dots + p(B|A_n)p(A_n)$$

$$p(x) = \frac{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}$$



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

$$S = v_0 t + \frac{a t^2}{2}$$

$$E = G \frac{m_1 m_2}{Q^2}$$

$$f(v) = A \exp\left(-\frac{mv}{15 \cdot 10^8}\right) \cdot \frac{1}{15 \cdot 10^8}$$



# Cointegration



$$D_x = \sigma^2 = M_x^2 - (M_x)^2$$

$$p_c(x) = \frac{1}{\sigma} e^{-x/\sigma}$$

$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^n p_i x_i$$

$$D_x = \sum_{i=1}^n p_i (x_i - M_x)^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = A \sqrt{\frac{k^3}{\pi}} \sqrt{v} e^{-kv^2}$$

$$v = \frac{2V}{n \sqrt{2\pi} d^2}$$

$$C = 4 \pi n v \frac{2V}{n \sqrt{2\pi} d^2}$$

$$C = \frac{\pi n S}{d}$$



$$d = \frac{h \cdot l}{2 \sin \alpha}$$

$$d^2 = d_1^2 + d_2^2 + 2 d_1 d_2 \cos(\varphi_2 - \varphi_1)$$

$$n v = A + \frac{m v^2}{2}$$

$$0,020 \cdot p(v)$$



$$E = m_0 c^2 + \frac{m v^2}{2}$$

$$m = m_0 \sqrt{1 - \beta^2}$$

$$S^2 = c^2 t^2 - l^2 = i n v$$

$$r_n = \frac{4 \pi \epsilon_0 n^2 a^2}{m Z e^2}$$

$$f(x) = \frac{e^x}{(x-1)^2}$$



$$D_x = \int_{-\infty}^{\infty} (x - M_x) f(x) dx$$

- Cointegration is the VAR version of unit roots
- Establishes long run relationships between two unit root variables
  - ▶ Consumption has a unit root, income has a unit root
  - ▶ Consumption - Income : ?????

## Definition (Integrated of Order 1)

A variable  $y_t$  is integrated of order 1 ( $I(1)$ ) if  $y_t$  is non-stationary and  $\Delta y_t = y_t - y_{t-1}$  is stationary.

$$R^2 = \frac{r^2}{(1-r^2)}$$

$$= I(1)$$

$$D_t = \int_{-\infty}^{\infty} (x - M_x / \sigma(x)) dx$$

## Definition (Bivariate Cointegration)

If  $x_t$  and  $y_t$  are cointegrated if both are  $I(1)$  and there exists a vector  $\beta$  with both elements non-zero such that

$$\beta_1 x_t - \beta_2 y_t \sim I(0)$$

- Strong link between  $x_t$  and  $y_t$
- Both are random walks but difference is mean reverting
- Mean reversion to the trend (stochastic trend)

# What does cointegration look like?

$$p_i = \int_{-\infty}^{\infty} (x - \mu_i)^2 g(x) dx$$

$$\mathbf{y}_t = \Phi_{ij} \mathbf{y}_{t-1} + \epsilon_t$$

$$\Phi_{11} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix}$$
$$\lambda_i = 1, 0.6$$

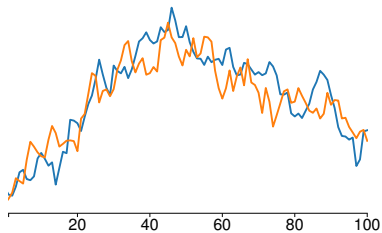
$$\Phi_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\lambda_i = 1, 1$$

$$\Phi_{21} = \begin{bmatrix} .7 & .2 \\ .2 & .7 \end{bmatrix}$$
$$\lambda_i = 0.9, 0.5$$

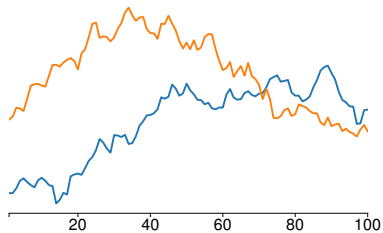
$$\Phi_{22} = \begin{bmatrix} -.3 & .3 \\ .1 & -.2 \end{bmatrix}$$
$$\lambda_i = -0.43, -0.06$$

# Persistence, Anti-persistence and Cointegration

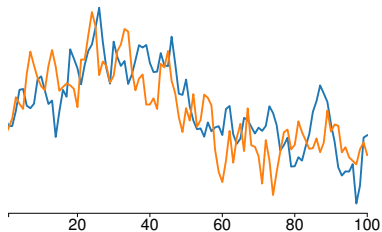
Cointegration ( $\Phi_{11}$ )



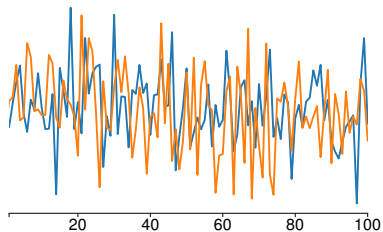
Independent Unit Roots ( $\Phi_{12}$ )



Persistent, Stationary ( $\Phi_{21}$ )



Anti-persistent, Stationary ( $\Phi_{22}$ )



# How do we know when a VAR is cointegrated?

- Eigenvalue condition determines whether a VAR(1) is cointegrated

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Cointegrated if only 1 eigenvalue is unity.
- If all less than 1: ?
- If both 1: two independent unit roots

$$\Phi_{11} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \quad \Phi_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda_i = 1, 0.6$                        $\lambda_i = 1, 1$

$$\Phi_{21} = \begin{bmatrix} .7 & .2 \\ .2 & .7 \end{bmatrix} \quad \Phi_{22} = \begin{bmatrix} -.3 & .3 \\ .1 & -.2 \end{bmatrix}$$

$\lambda_i = 0.9, 0.5$                        $\lambda_i = -0.43, -0.06$

$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$$

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$$A_n^k = \frac{n!}{(n-k)!}$$

$$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

$$\lambda_n^k = \underbrace{n \cdot n \cdot \dots \cdot n}_k = n^k$$

$$\tilde{C}_n^m = P_{m, n-m-1} = \frac{(n+m-1)!}{m!(n-1)!}$$

$$\tilde{P}_{n_1, n_2, \dots, n_k} = \frac{(n_1 + n_2 + \dots + n_k)!}{n_1! n_2! \dots n_k!}$$

$$C_n^k = \frac{n!}{k!(n-k)!}$$

$$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + p(B|A_3)p(A_3) + \dots + p(B|A_n)p(A_n)$$

$$p(x) = \frac{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}$$



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$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

$$S = \sqrt{\sigma^2} = \frac{d^2}{2}$$

$$E = G \frac{m_1 m_2}{Q^2}$$

$$f(x) = A \exp\left(-\frac{m_1}{15x^2}\right) \exp\left(-\frac{m_2}{15x^2}\right)$$



# Error Correction Models



$$D_x = \sigma^2 = M_x^2 - (M_x)^2$$

$$P_x(k) = \frac{1}{k!} e^{-x} x^k$$

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^n p_i x_i$$

$$D_x = \sum_{i=1}^n p_i (x_i - M_x)^2$$

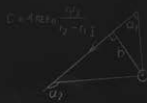
$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = A \sqrt{\frac{k^3}{\pi}} \sqrt{v} e^{-kv^2}$$

$$C = \frac{2V}{\pi \sqrt{2\pi} d^2}$$

$$C = \frac{\pi x S}{d}$$



$$A = \frac{A_1}{\sin \varphi_2} = \frac{A_2}{\sin \varphi_1}$$

$$A^2 = A_1^2 + A_2^2 + 2 A_1 A_2 \cos(\varphi_2 - \varphi_1)$$

$$nV = A + \frac{mV^2}{2}$$

$$0,020 \rho(v)$$



$$E = m_0 c^2 + \frac{mv^2}{2}$$

$$m = m_0 \sqrt{1 - \beta^2}$$

$$S^2 = c^2 t^2 - l^2 = in v$$

$$r_n = \frac{4\pi \epsilon_0 n^2 a^2}{m^2 Z e^2}$$



# Error Correction Models

- Major point of cointegration
  - ▶ Cointegrated  $\Leftrightarrow$  Error correction model
- What is an error correction model?

- ▶ Cointegrated VAR:

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- ▶ Error correction model:

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 & .2 \\ .2 & -.2 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- ▶ Normalized form

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 \\ .2 \end{bmatrix} [1 \quad -1] \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- $[1 \quad -1]$  is cointegrating vector
- $[-.2 \quad .2]'$  measures the speed of adjustment

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

Subtracting  $[y_{t-1} \ x_{t-1}]'$  from both sides

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} - \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} - \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \left( \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 & .2 \\ .2 & -.2 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 \\ .2 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

# Cointegrating vectors

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 & .2 \\ .2 & -.2 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$
$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 \\ .2 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Cointegrating relationship can always be decomposed

$$\Delta \mathbf{y}_t = \boldsymbol{\pi} \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{\pi} = \boldsymbol{\alpha} \boldsymbol{\beta}'$$

- $\boldsymbol{\alpha}$  measures the speed of convergence
- $\boldsymbol{\beta}$  contain the cointegrating vectors
- Number of cointegrating vectors is  $\text{rank}(\boldsymbol{\alpha} \boldsymbol{\beta}')$

$$\boldsymbol{\alpha} \boldsymbol{\beta}' = \begin{bmatrix} 0.3 & 0.2 & -0.36 \\ 0.2 & 0.5 & -0.35 \\ -0.3 & -0.3 & 0.39 \end{bmatrix}$$

- How many?

# Determining the cointegrating vectors

$$\hat{\beta}_1 = \begin{bmatrix} 1 & -M_2 / \alpha_1 \alpha_2 \end{bmatrix}$$

$$\Delta \mathbf{y}_t = \boldsymbol{\pi} \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{\pi} = \begin{bmatrix} 0.3 & 0.2 & -0.36 \\ 0.2 & 0.5 & -0.35 \\ -0.3 & -0.3 & 0.39 \end{bmatrix}$$

- Put  $\boldsymbol{\pi}$  in row echelon form

$$\text{Row Echelon Form} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Recall  $\boldsymbol{\pi} = \boldsymbol{\alpha} \boldsymbol{\beta}'$

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -0.3 \end{bmatrix} \quad \boldsymbol{\alpha} = \begin{bmatrix} .3 & .2 \\ .2 & .5 \\ -0.3 & -0.3 \end{bmatrix}$$

# Solving for the cointegrating vectors

$$\beta_2 = \begin{bmatrix} \alpha_1 - \alpha_2 / \alpha_{11} \\ \alpha_2 \end{bmatrix}$$

$$\alpha\beta' = \begin{bmatrix} 0.3 & 0.2 & -0.36 \\ 0.2 & 0.5 & -0.35 \\ -0.3 & -0.3 & 0.39 \end{bmatrix}$$

$$\text{Row-Echelon Form} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \beta_1 & \beta_2 \end{bmatrix}$$

and  $\alpha$  has 6 unknown parameters.  $\alpha\beta'$  can be combined to produce

$$\pi = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{11}\beta_1 + \alpha_{12}\beta_2 \\ \alpha_{21} & \alpha_{22} & \alpha_{21}\beta_1 + \alpha_{22}\beta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{31}\beta_1 + \alpha_{32}\beta_2 \end{bmatrix}$$

$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$$

$$P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

$$A_n^k = \frac{n!}{(n-k)!}$$

$$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

$$\lambda_n^k = \underbrace{n \cdot n \cdot \dots \cdot n}_k = n^k$$

$$\tilde{C}_n^m = P_{m, n-1} = \frac{(n+m-1)!}{m!(n-1)!}$$

$$\tilde{P}_{n_1, n_2, \dots, n_k} = \frac{(n_1 + n_2 + \dots + n_k)!}{n_1! n_2! \dots n_k!}$$

$$C_n^c = \frac{n!}{c!(n-c)!}$$

$$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + p(B|A_3)p(A_3) + \dots + p(B|A_n)p(A_n)$$

$$p(x) = \frac{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}$$



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

$$S = \sqrt{\sigma^2} = \frac{d^2}{2}$$

$$E = G \frac{m_1 m_2}{Q^2}$$

$$f(x) = A \exp\left(-\frac{m_1}{15x^2}\right) \exp\left(-\frac{m_2}{15x^2}\right)$$



# Testing for Cointegration



$$D_x = \sigma^2 = M_{x^2} - (M_x)^2$$

$$P_c(x) = \frac{1}{\sigma} e^{-x/\sigma}$$

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^n p_i x_i$$

$$D_x = \sum_{i=1}^n p_i (x_i - M_x)^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = A \sqrt{\frac{k^3}{\pi}} \sqrt{v} e^{-kv^2}$$

$$C = \frac{2V}{\pi \sqrt{2\pi} d^2}$$

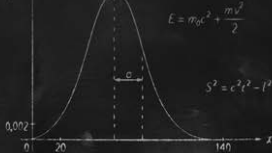


$$a = \frac{a_1}{\sin \phi} = \frac{a_2}{\cos \phi}$$

$$a^2 = a_1^2 + a_2^2 + 2 a_1 a_2 \cos(\phi_2 - \phi_1)$$

$$nv = A + \frac{mv^2}{2}$$

$$0.020 \rho(v)$$



$$E = m_0 c^2 + \frac{m v^2}{2}$$

$$m = m_0 \sqrt{1 - \beta^2}$$

$$S^2 = c^2 t^2 - l^2 = in v$$

$$r_n = \frac{4\pi \epsilon_0 n^2 n^2}{m^2 c^2}$$

# Testing for Cointegration



$$\sigma_y^2 = \int_{-\infty}^{\infty} (x - \mu_y)^2 \phi(x) dx$$

- Two tests for cointegration
  - ▶ Engle-Granger
  - ▶ Johansen
- We will focus on Engle-Granger
  - ▶ Simple and intuitive
  - ▶ Only applicable with 1 cointegrating relationship
- Test key property of cointegration: **difference is I(0)**
- Most of the work is a simple OLS

$$y_t = \delta_0 + \beta x_t + \epsilon_t$$

- Rest of work is testing  $\hat{\epsilon}_t$  for a unit root
- Johansen tests eigenvalues of  $\pi = \alpha\beta'$  directly.

## Algorithm (Engle-Granger Test)

1. *Begin by analyzing  $x_t$  and  $y_t$  in isolation. Both must be unit roots to consider cointegration.*
2. *Estimate the long run relationship*

$$y_t = \delta_0 + \beta x_t + \epsilon_t$$

*and test  $H_0 : \gamma = 0$  against  $H_0 : \gamma < 0$  in the ADF regression*

$$\Delta \hat{\epsilon}_t = \gamma \hat{\epsilon}_{t-1} + \delta_1 \Delta \hat{\epsilon}_{t-1} + \dots + \delta_p \Delta \hat{\epsilon}_{t-p} + \eta_t.$$

3. *Using the estimated parameters, specify and estimate the error correction form of the relationship,*

$$\begin{bmatrix} \Delta x_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} \pi_{01} \\ \pi_{02} \end{bmatrix} + \begin{bmatrix} \alpha_1 \hat{\epsilon}_t \\ \alpha_2 \hat{\epsilon}_t \end{bmatrix} + \boldsymbol{\pi}_1 \begin{bmatrix} \Delta x_{t-1} \\ \Delta y_{t-1} \end{bmatrix} + \dots + \boldsymbol{\pi}_P \begin{bmatrix} \Delta x_{t-P} \\ \Delta y_{t-P} \end{bmatrix} + \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix}$$

4. *Assess the model*



## ■ Deterministic terms

- ▶ No deterministic terms: only in special circumstances

$$y_t = \beta x_t + \epsilon_t$$

- ▶ Constant: standard case

$$y_t = \delta_0 + \beta x_t + \epsilon_t$$

- ▶ Time trend and constant: allow different growth rates/time trends in variables

$$y_t = \delta_0 + \delta_1 t + \beta x_t + \epsilon_t$$

## ■ Critical Values

- ▶ Critical values depend on the deterministics in the CI regression
  - Models with more deterministics have lower (more negative) critical values
- ▶ Critical values depend on number of RHS  $I(1)$  variables
  - Larger models have lower critical values

# Example: *cay*

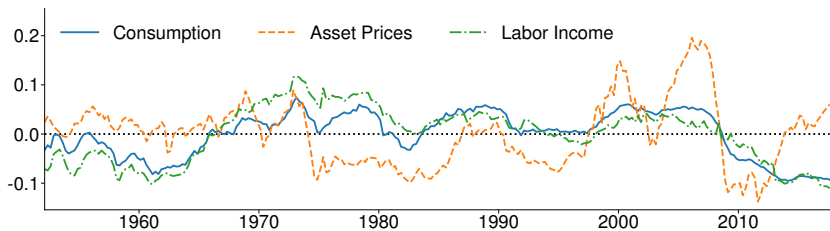
- Consumption-Aggregate Wealth has been an interesting cointegrated series in recent finance literature
- Has revived the CCAPM
- Three components:
  - ▶ Consumption ( $c$ )
  - ▶ Asset Wealth ( $a$ )
  - ▶ Labor Income (Human Wealth) ( $y$ )
- Deviation from long run related to expected return
- Cointegrating relationship:  $c_t + .643 - 0.249a_t - 0.785y_t$

## Unit Root Tests

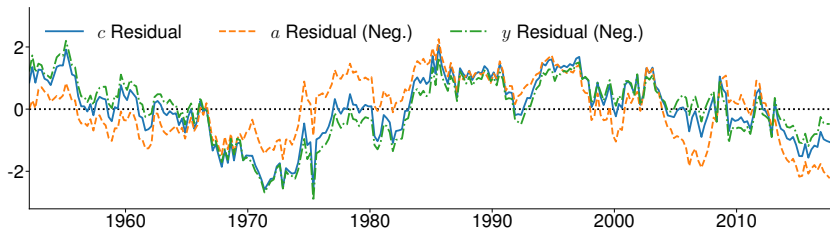
Series	T-stat	P-val	ADF Lags
$c$	-1.198	0.674	5
$a$	-0.205	0.938	3
$y$	-2.302	0.171	0
$\hat{\epsilon}_t^c$	-2.706	0.383	1
$\hat{\epsilon}_t^a$	-2.573	0.455	0
$\hat{\epsilon}_t^y$	-2.679	0.398	1

$$\hat{\beta}_1 = (X'X)^{-1}X'Y$$

## Original Series (logs)



## Error



# Vector Error Correction Model

- VECM estimated using the residuals from cointegrating regression

$$\begin{bmatrix} \Delta c_t \\ \Delta a_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} 0.003 \\ (0.000) \\ 0.004 \\ (0.014) \\ 0.003 \\ (0.000) \end{bmatrix} + \begin{bmatrix} -0.000 \\ (0.281) \\ 0.002 \\ (0.037) \\ 0.000 \\ (0.515) \end{bmatrix} \hat{\epsilon}_{t-1} + \begin{bmatrix} 0.192 & 0.102 & 0.147 \\ (0.005) & (0.000) & (0.004) \\ 0.282 & 0.220 & -0.149 \\ (0.116) & (0.006) & (0.414) \\ 0.369 & 0.061 & -0.139 \\ (0.000) & (0.088) & (0.140) \end{bmatrix} \begin{bmatrix} \Delta c_{t-1} \\ \Delta a_{t-1} \\ \Delta y_{t-1} \end{bmatrix} + \eta_t$$

- P-values in parentheses
- Estimation of cointegration relationship has no effect on standard errors
  - ▶ Converges fast ( $T$ )
  - ▶ VECM parameters converge at rate  $\sqrt{T}$

$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$$

$$P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

$$A_n^k = \frac{n!}{(n-k)!}$$

$$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

$$\lambda_n^k = \underbrace{n \cdot n \cdot \dots \cdot n}_k = n^k$$

$$\tilde{C}_n^m = P_{m, n-m-1} = \frac{(n+m-1)!}{m!(n-1)!}$$

$$\tilde{C}_n^{r_1, r_2, \dots, r_k} = \frac{(n_1 + n_2 + \dots + n_k)!}{n_1! n_2! \dots n_k!}$$

$$C_n^c = \frac{n!}{c!(n-c)!}$$

$$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + p(B|A_3)p(A_3) + \dots + p(B|A_n)p(A_n)$$

$$p(x) = \frac{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}$$



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$V_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

$$S = v\phi + \frac{d^2}{2}$$

$$E = G \frac{m_1 m_2}{Q^2}$$

$$f(v) = 4\pi \left(\frac{mv}{15c^2}\right)^3 \cdot 2v \cdot \frac{d^2}{2}$$



# Spurious Regression



$$D_x = \sigma^2 = M_x^2 - (M_x)^2$$

$$p_c(x) = \frac{1}{\sigma} e^{-x/\sigma}$$

$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^c p_i x_i$$

$$D_x = \sum_{i=1}^c p_i (x_i - M_x)^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi c}} e^{-\frac{(x-a)^2}{2c}}$$

$$\phi(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = 4 \sqrt{\frac{k^3}{\pi}} \sqrt{v} e^{-kv^2}$$

$$v = \frac{2V}{n \sqrt{2\pi} d^2}$$

$$C = 4 \pi n v \frac{2V}{n \sqrt{2\pi} d^2}$$

$$C = \frac{\pi n S}{d}$$

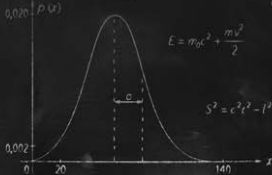


$$a = \frac{a_1}{\sin \phi} = \frac{a_2}{\cos \phi}$$

$$a^2 = a_1^2 + a_2^2 + 2 a_1 a_2 \cos(\phi_2 - \phi_1)$$

$$nv = A + \frac{mv^2}{2}$$

$$0.020 \rho(v)$$



$$E = m v_c^2 + \frac{m v^2}{2}$$

$$m = m_0 \sqrt{1 - \beta^2}$$

$$S^2 = c^2 t^2 - l^2 = in v$$

$$r_n = \frac{4\pi \epsilon_0 n^2 n^2}{m^2 c^2}$$

# Spurious Regression and Balance

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

- Caution is needed when working with I(1) data
  - ▶ I(0) on I(0): The usual case. Standard asymptotic arguments apply.
  - ▶ I(1) on I(0): This regression is unbalanced.
  - ▶ I(1) on I(1): Cointegration or spurious regression.
  - ▶ I(0) on I(1): This regression is unbalanced.
- Spurious regression can lead to large  $t$ -stats when the series are independent.
  - ▶ Two unrelated I(1) processes,  $x_t$  and  $y_t$

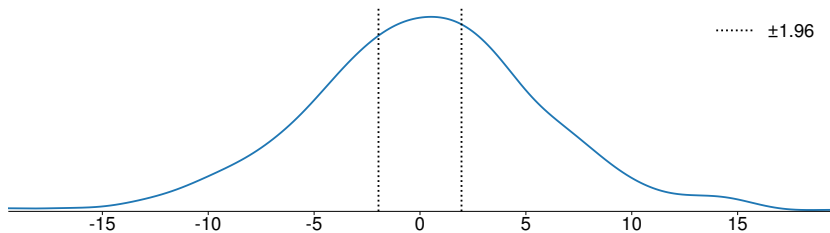
$$x_t = x_{t-1} + \epsilon_t$$

$$y_t = y_{t-1} + \eta_t$$

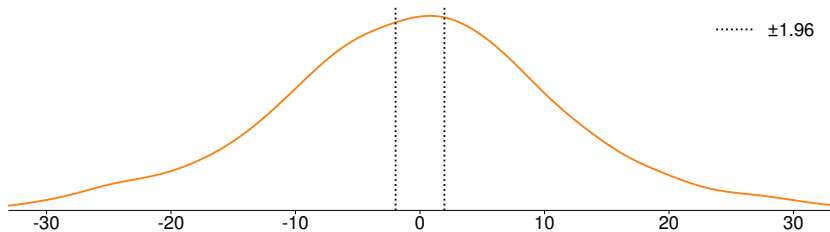
- ▶ When  $T = 50$ , approx 80% of  $t$ -stats are significant
- ▶ Always check for I(1) when using time-series data
- ▶ If both I(1), make sure cointegrated.

# Spurious Regression

$T = 50$



$T = 200$



$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$$

$$P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

$$A_n^k = \frac{n!}{(n-k)!}$$

$$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

$$\lambda_n^k = \underbrace{n \cdot n \cdot \dots \cdot n}_k = n^k$$

$$\tilde{C}_n^m = P_{m, n-1} = \frac{(n+m-1)!}{m!(n-1)!}$$

$$\tilde{C}_n^{r_1, r_2, \dots, r_k} = \frac{(n_1 + n_2 + \dots + n_k)!}{n_1! n_2! \dots n_k!}$$

$$C_n^c = \frac{n!}{c!(n-c)!}$$

$$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + p(B|A_3)p(A_3) + \dots + p(B|A_n)p(A_n)$$

$$p(x) = \frac{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}$$



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

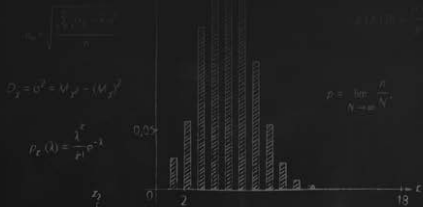
$$S = \sqrt{\sigma^2} = \frac{\sigma^2}{2}$$

$$E = G \frac{m_1 m_2}{Q^2}$$

$$f(x) = A \exp\left(-\frac{m_1}{2\sigma^2} x^2\right) \exp\left(-\frac{m_2}{2\sigma^2} x^2\right)$$



# Revisiting Cross-Sectional Regression



$$D_x = \sigma^2 = M_{x^2} - (M_x)^2$$

$$p_c(x) = \frac{1}{\sigma} e^{-x/\sigma}$$

$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^n p_i x_i$$

$$D_x = \sum_{i=1}^n p_i (x_i - M_x)^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-M_x)^2}{2\sigma^2}}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = A \sqrt{\frac{k^3}{\pi}} \sqrt{v} e^{-kv^2}$$

$$C = \frac{2V}{\pi \sqrt{2\pi} d^2}$$

$$C = \frac{\pi k S}{d}$$



$$a = \frac{a_1}{\sin \varphi_1} = \frac{a_2}{\cos \varphi_1}$$

$$a^2 = a_1^2 + a_2^2 + 2A_1 A_2 \cos(\varphi_1 + \varphi_2)$$

$$nv = A + \frac{mv^2}{2}$$

$$0.020 \rho(v)$$



$$E = \sigma_0^2 c^2 + \frac{m^2}{2}$$

$$m = m_0 \sqrt{1 - \zeta^2}$$

$$S^2 = c^2 t^2 - l^2 = in v$$

$$r_n = \frac{4\pi \epsilon_0 n^2 n^2}{m^2 c^2}$$



# Cross-section Regression with Time Series Data

- It is common to run regressions using time-series data

$$y_t = \mathbf{x}_t \boldsymbol{\beta} + \epsilon_t$$

- Using time-series data in a cross-sectional regression may require modification to inference
- Modification is needed if the scores  $\{\mathbf{x}_t \epsilon_t\}$  are autocorrelated

$$\begin{aligned} \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} &= \left( \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \epsilon_t \\ \Rightarrow V \left[ \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \right] &\approx \boldsymbol{\Sigma}_{\mathbf{X}\mathbf{X}}^{-1} V \left[ \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \epsilon_t \right] \boldsymbol{\Sigma}_{\mathbf{X}\mathbf{X}}^{-1} \end{aligned}$$

- ▶ Usually occurs when the errors  $\epsilon_t$  are autocorrelated due to mis- or under-specification of the model

# Why the difference?

- Consider the estimation of the mean when  $y_t$  has white noise errors

$$y_t = \mu + \epsilon_t$$

- Obviously
- The sample mean is unbiased

$$\begin{aligned} \mathbb{E}[\hat{\mu}] &= \mathbb{E} \left[ T^{-1} \sum_{t=1}^T y_t \right] \\ &= T^{-1} \sum_{t=1}^T \mathbb{E} [y_t] \\ &= \mu \end{aligned}$$

# Why the difference?

- The variance of the sample mean

$$\begin{aligned}V[\hat{\mu}] &= \mathbb{E} \left[ \left( T^{-1} \sum_{t=1}^T y_t - \mu \right)^2 \right] \\&= \mathbb{E} \left[ T^{-2} \left( \sum_{t=1}^T \epsilon_t^2 + \sum_{r=1}^T \sum_{s=1, r \neq s}^T \epsilon_r \epsilon_s \right) \right] \\&= T^{-2} \sum_{t=1}^T \mathbb{E}[\epsilon_t^2] + T^{-2} \sum_{r=1}^T \sum_{s=1, r \neq s}^T \mathbb{E}[\epsilon_r \epsilon_s] \\&= T^{-2} \sum_{t=1}^T \sigma^2 + T^{-2} \sum_{r=1}^T \sum_{s=1, r \neq s}^T 0 \\&= \frac{\sigma^2}{T},\end{aligned}$$

- Due to white noise,  $\mathbb{E}[\epsilon_i \epsilon_j] = 0$  whenever  $i \neq j$ .
- This is the usual result

# The case of an MA(1) error



- Now suppose that the error follows an MA(1)

$$\eta_t = \theta\epsilon_{t-1} + \epsilon_t$$

where  $\{\epsilon_t\}$  is a white noise process

- Error is mean 0 and so sample mean is still unbiased
- Variance of sample mean is *different* since  $\eta_t$  is autocorrelated
  - $E[\eta_t\eta_{t-1}] \neq 0$ .

$$\begin{aligned} V[\hat{\mu}] &= E \left[ \left( T^{-1} \sum_{t=1}^T \eta_t \right)^2 \right] \\ &= E \left[ T^{-2} \left( \sum_{t=1}^T \eta_t^2 + 2 \sum_{t=1}^{T-1} \eta_t \eta_{t+1} + 2 \sum_{t=1}^{T-2} \eta_t \eta_{t+2} + \dots + \right. \right. \\ &\quad \left. \left. 2 \sum_{t=1}^2 \eta_t \eta_{t+T-2} + 2 \sum_{t=1}^1 \eta_t \eta_{t+T-1} \right) \right] \end{aligned}$$

# The case of an MA(1) error

$$\eta_t = \epsilon_t + \theta \epsilon_{t-1}$$

- In terms of autocovariances,

$$\begin{aligned} V[\hat{\mu}] &= T^{-2} \sum_{t=1}^T E[\eta_t^2] + 2T^{-2} \sum_{t=1}^{T-1} E[\eta_t \eta_{t+1}] + 2T^{-2} \sum_{t=1}^{T-2} E[\eta_t \eta_{t+2}] + \dots + \\ &\quad 2T^{-2} \sum_{t=1}^2 E[\eta_t \eta_{t+T-2}] + 2T^{-2} \sum_{t=1}^1 E[\eta_t \eta_{t+T-1}] \\ &= T^{-2} \sum_{t=1}^T \gamma_0 + 2T^{-2} \sum_{t=1}^{T-1} \gamma_1 + 2T^{-2} \sum_{t=1}^{T-2} \gamma_2 + \dots + 2T^{-2} \sum_{t=1}^1 \gamma_{T-1} \end{aligned}$$

- $\gamma_0 = V[\eta_t] = (1 + \theta^2) V[\epsilon_t]$  and  $\gamma_s = E[\eta_t \eta_{t-s}]$
- An MA(1) has 1 non-zero autocovariance,

$$\begin{aligned} \gamma_1 &= E[\eta_t \eta_{t-1}] \\ &= E[(\theta \epsilon_{t-1} + \epsilon_t)(\theta \epsilon_{t-2} + \epsilon_{t-1})] \\ &= \theta^2 E[\epsilon_{t-1} \epsilon_{t-2}] + \theta E[\epsilon_{t-1}^2] + \theta E[\epsilon_t \epsilon_{t-2}] + E[\epsilon_t \epsilon_{t-1}] \\ &= \theta \sigma^2 \end{aligned}$$

# The case of an MA(1) error

- Putting it all together

$$\begin{aligned}V[\hat{\mu}] &= T^{-2} \sum_{t=1}^T \gamma_0 + 2T^{-2} \sum_{t=1}^{T-1} \gamma_1 \\&= T^{-2}T\gamma_0 + 2T^{-2}(T-1)\gamma_1 \\&\approx \frac{\gamma_0 + 2\gamma_1}{T} \\&= \frac{\sigma^2(1 + \theta^2 + 2\theta)}{T}\end{aligned}$$

Can be larger or smaller ( $-2 < \theta < 0$ )

The variance of the sum is the sum of the variance  
*only* when the errors are uncorrelated

# Estimating the parameter covariance (from CS lecture)

- When the scores are uncorrelated (a Martingale Difference sequence (MDS)) White's covariance estimator is consistent

## Theorem (Consistency of Asymptotic Covariance Estimator)

*Under the large sample assumptions,*

$$\hat{\Sigma}_{\mathbf{X}\mathbf{X}} = T^{-1} \mathbf{X}'\mathbf{X} \xrightarrow{p} \Sigma_{\mathbf{X}\mathbf{X}}$$

$$\hat{\mathbf{S}} = T^{-1} \sum_{t=1}^T \hat{\epsilon}_t^2 \mathbf{x}_t' \mathbf{x}_t \xrightarrow{p} \mathbf{S}$$

*and*

$$\hat{\Sigma}_{\mathbf{X}\mathbf{X}}^{-1} \hat{\mathbf{S}} \hat{\Sigma}_{\mathbf{X}\mathbf{X}}^{-1} \xrightarrow{p} \Sigma_{\mathbf{X}\mathbf{X}}^{-1} \mathbf{S} \Sigma_{\mathbf{X}\mathbf{X}}^{-1}$$

# Modification to regression parameter covariance

- White's estimator is only heteroskedasticity robust – not heteroskedasticity and autocorrelation robust

$$\hat{\mathbf{S}} = T^{-1} \sum_{t=1}^T \hat{\epsilon}_t^2 \mathbf{x}_t' \mathbf{x}_t \xrightarrow{p} \mathbf{S}$$

- Solution is to use a Newey-West covariance for the scores ( $\mathbf{x}_t \epsilon_t$ )

## Definition (Newey-West Covariance Estimator)

Let  $\mathbf{z}_t$  be a  $k$  by 1 vector series that may be autocorrelated and define  $\mathbf{z}_t^* = \mathbf{z}_t - \bar{\mathbf{z}}$  where  $\bar{\mathbf{z}} = T^{-1} \sum_{t=1}^T \mathbf{z}_t$ . The  $L$ -lag Newey-West covariance estimator for the variance of  $\bar{\mathbf{z}}$  is

$$\hat{\Sigma}_{NW} = \hat{\Gamma}_0 + \sum_{l=1}^L w_l (\hat{\Gamma}_l + \hat{\Gamma}_l')$$

where  $\hat{\Gamma}_l = T^{-1} \sum_{t=l+1}^T \mathbf{z}_t^* \mathbf{z}_{t-l}^{*'} and  $w_l = 1 - \frac{l}{L+1}$ .$



# Modification to regression parameter covariance

- Applied to a cross-sectional regression with time-series data

$$\begin{aligned}\hat{\mathbf{S}}_{NW} &= T^{-1} \left( \sum_{t=1}^T e_t^2 \mathbf{x}'_t \mathbf{x}_t + \sum_{l=1}^L w_l \left( \sum_{s=l+1}^T e_s e_{s-l} \mathbf{x}'_s \mathbf{x}_{s-l} + \sum_{q=l+1}^T e_{q-l} e_q \mathbf{x}'_{q-l} \mathbf{x}_q \right) \right) \\ &= \hat{\mathbf{\Gamma}}_0 + \sum_{l=1}^L w_l (\hat{\mathbf{\Gamma}}_l + \hat{\mathbf{\Gamma}}_l')\end{aligned}$$

- The HAC robust covariance of  $\hat{\beta}$  is

$$\hat{\Sigma}_{\mathbf{X}\mathbf{X}}^{-1} \hat{\mathbf{S}}_{NW} \hat{\Sigma}_{\mathbf{X}\mathbf{X}}^{-1}$$

# Considerations when using Newey-West an estimator

- Is a Newey-West estimator needed? **Complex estimators have worse finite sample performance**
- It **must** be the case that  $L \rightarrow \infty$  as  $T \rightarrow \infty$
- Even if the scores follow a MA(1)!
- Optimal rate is  $O(T^{\frac{1}{3}})$  so  $L \propto T^{\frac{1}{3}}$  or  $L = cT^{\frac{1}{3}}$  for some (unknown)  $c$
- Other HAC estimators available and may work well if the scores very persistent
  - ▶ Den Haan-Levin
- Alternative is to include lagged regressand(s) in the regression

$$y_t = \mathbf{x}_t\boldsymbol{\beta} + \sum_{p=1}^P \phi_p y_{t-p} + \epsilon_t$$

- ▶ Not popular when focus is on cross-section component of model









