Analysis of Multiple Time Series

Kevin Sheppard
http://www.kevinsheppard.com

Oxford MFE

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This week's material

- Vector Autoregressions
- Basic examples
- Properties
  - Stationarity
- Revisiting univariate ARMA processes
- Forecasting
  - Granger Causality
  - Impulse Response functions
- Cointegration
  - Examining long-run relationships
  - Determining whether a VAR is cointegrated
  - Error Correction Models
  - Testing for Cointegration
    - Engle-Granger

Lots of revisiting univariate time series.

Vector Autoregressions

Why VAR analysis?

- Stationary VARs
  - Determine whether variables feedback into one another
  - Improve forecasts
  - Model the effect of a shock in one series on another
  - Differentiate between short-run and long-run dynamics
- Cointegration
  - Link random walks
  - Uncover long run relationships
  - Can improve medium to long term forecasting a lot

VAR Defined

- P^{th} order autoregression, AR(P):

$$y_t = \phi_0 + \phi_1 y_{t-1} + \ldots + \phi_P y_{t-p} + \epsilon_t$$

- P^{th} order vector autoregression, VAR(P):

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \ldots + \mathbf{\Phi}_P \mathbf{y}_{t-p} + \boldsymbol{\epsilon}_t$$

  where $\mathbf{y}_t$ and $\boldsymbol{\epsilon}_t$ are k by 1 vectors

- Bivariate VAR(1):

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Compactly expresses two linked models:

$$y_{1,t} = \phi_{01} + \phi_{11} y_{1,t-1} + \phi_{12} y_{2,t-1} + \epsilon_{1,t}$$

$$y_{2,t} = \phi_{02} + \phi_{21} y_{1,t-1} + \phi_{22} y_{2,t-1} + \epsilon_{2,t}$$

Stationarity Revisited

- Stationarity is a statistically meaningful form of regularity. A stochastic process $\{y_t\}$ is covariance stationary if

$$\begin{aligned} \mathrm{E}[y_t] &= \mu && \forall t \\ \mathrm{V}[y_t] &= \sigma^2 && \sigma^2 < \infty \forall t \\ \mathrm{E}[(y_t - \mu)(y_{t-s} - \mu)] &= \gamma_s && \forall t, s \end{aligned}$$

- AR(1) stationarity: $y_t = \phi y_{t-1} + \epsilon_t$
  - $|\phi| < 1$
  - ϵ_t is white noise
- AR(P) stationarity: $y_t = \phi_1 y_{t-1} + \ldots + \phi_P y_{t-P} + \epsilon_t$
  - Roots of $(z^P - \phi_1 z^{P-1} - \phi_2 z^{P-2} - \ldots - \phi_{P-1} z - \phi_P)$ less than 1
  - ϵ_t is white noise
- No dependence on t

Relationship to AR

- AR(1)

$$
\begin{aligned}
y_t &= \phi_0 + \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_0 + \phi_1(\phi_0 + \phi_1 y_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
&= \phi_0 + \phi_1\phi_0 + \phi_1^2 y_{t-2} + \phi_1\epsilon_{t-1} + \epsilon_t \\
&= \phi_0 + \phi_1\phi_0 + \phi_1^2(\phi_0 + \phi_1 y_{t-3} + \epsilon_{t-2}) + \phi_1\epsilon_{t-1} + \epsilon_t \\
&= \phi_0 \sum_{i=0}^{\infty} \phi_1^i + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i} \\
&= (1-\phi_1)^{-1}\phi_0 + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i}
\end{aligned}
$$

Relationship to AR

- VAR(1)

$$
\begin{aligned}
\mathbf{y}_t &= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t \\
&= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \big(\mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-2} + \boldsymbol{\epsilon}_{t-1}\big) + \boldsymbol{\epsilon}_t \\
&= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{\Phi}_0 + \mathbf{\Phi}_1^2 \mathbf{y}_{t-2} + \mathbf{\Phi}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\epsilon}_t \\
&= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{\Phi}_0 + \mathbf{\Phi}_1^2 \big(\mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-3} + \boldsymbol{\epsilon}_{t-2}\big) + \mathbf{\Phi}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\epsilon}_t \\
&= \sum_{i=0}^{\infty} \mathbf{\Phi}_1^i \mathbf{\Phi}_0 + \sum_{i=0}^{\infty} \mathbf{\Phi}_1^i \boldsymbol{\epsilon}_{t-i} \\
&= (\mathbf{I}_k - \mathbf{\Phi}_1)^{-1} \mathbf{\Phi}_0 + \sum_{i=0}^{\infty} \mathbf{\Phi}_1^i \boldsymbol{\epsilon}_{t-i}
\end{aligned}
$$

Properties of a VAR(1) and AR(1)

$$\text{AR(1)}: y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$
$$\text{VAR(1)}: \mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

	AR(1)	VAR(1)
Mean	$\phi_0/(1-\phi_1)$	$(\mathbf{I}_k - \mathbf{\Phi}_1)^{-1}\mathbf{\Phi}_0$
Variance	$\sigma^2/(1-\phi_1^2)$	$(\mathbf{I} - \mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1)^{-1}\text{vec}(\mathbf{\Sigma})$
sth Autocovariance	$\gamma_s = \phi_1^s \mathrm{V}[y_t]$	$\mathbf{\Gamma}_s = \mathbf{\Phi}_1^s \mathrm{V}[\mathbf{y}_t]$
-sth Autocovariance	$\gamma_{-s} = \phi_1^s \mathrm{V}[y_t]$	$\mathbf{\Gamma}_{-s} = \mathrm{V}[\mathbf{y}_t]\mathbf{\Phi}_1^{s\prime}$

Autocovariances of vector processes are not symmetric, but $\mathbf{\Gamma}_s = \mathbf{\Gamma}'_{-s}$

- Stationarity
  - AR(1): $|\phi_1| < 1$
  - VAR(1): $|\lambda_i| < 1$ where λ_i are the eigenvalues of $\mathbf{\Phi}_1$

Stock and Bond VAR

- VWM from CRSP
- TERM constructed from 10-year bond *return minus 1-year return* from FRED
- February 1962 until December 2018 (683 months)

$$\begin{bmatrix} VWM_t \\ TERM_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11,1} & \phi_{12,1} \\ \phi_{21,1} & \phi_{22,1} \end{bmatrix} \begin{bmatrix} VWM_{t-1} \\ TERM_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Market model:

$$VWM_t = \phi_{01} + \phi_{11,1} VWM_{t-1} + \phi_{12,1} 10YR_{t-1} + \epsilon_{1,t}$$

- Long bond model

$$TERM_t = \phi_{01} + \phi_{21,1} VWM_{t-1} + \phi_{22,1} TERM_{t-1} + \epsilon_{2,t}.$$

- Estimates

$$\begin{bmatrix} VWM_t \\ TERM_t \end{bmatrix} = \begin{bmatrix} \underset{(0.000)}{0.801} \\ \underset{(0.041)}{0.232} \end{bmatrix} + \begin{bmatrix} \underset{(0.122)}{0.059} & \underset{(0.004)}{0.166} \\ \underset{(0.000)}{-0.104} & \underset{(0.002)}{0.116} \end{bmatrix} \begin{bmatrix} VWM_{t-1} \\ TERM_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

Stock and Bond VAR

- Estimates from VAR

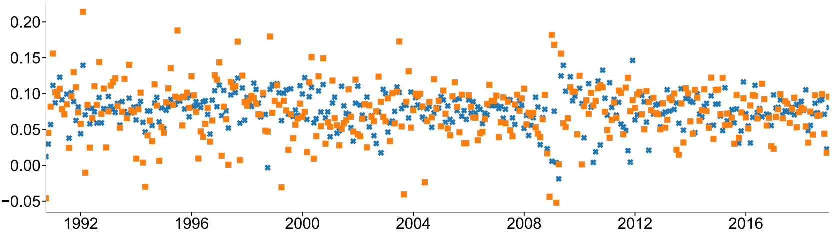
$$
\begin{aligned}
VWM_t &= \underset{(0.000)}{0.816} + \underset{(0.117)}{0.060}\, VWM_{t-1} + \underset{(0.003)}{0.168}\, TERM_{t-1} \\
TERM_t &= \underset{(0.045)}{0.228} - \underset{(0.000)}{0.104}\, VWM_{t-1} + \underset{(0.002)}{0.115}\, TERM_{t-1}
\end{aligned}
$$

- Estimates from AR

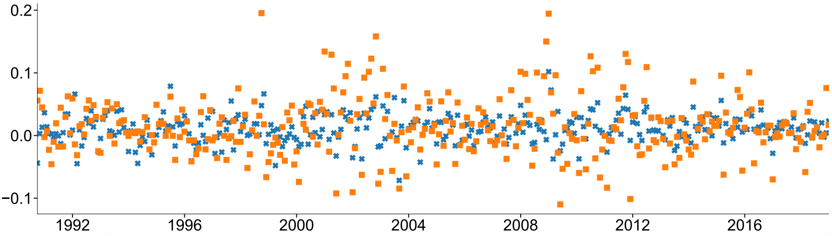
$$
\begin{aligned}
VWM_t &= \underset{(0.000)}{0.830} + \underset{(0.057)}{0.073}\, VWM_{t-1} \\
TERM_t &= \underset{(0.224)}{0.137} + \underset{(0.011)}{0.098}\, TERM_{t-1}
\end{aligned}
$$

Comparing AR and VAR forecasts

1-month-ahead forecasts of the VWM returns



1-month-ahead forecasts of 10-year bond returns



Monetary Policy VAR

- Standard tool in monetary policy analysis
  - Unemployment rate (differenced)
  - Federal Funds rate
  - Inflation rate (differenced)

$$\begin{bmatrix} \Delta\text{UNEMP}_t \\ \text{FF}_t \\ \Delta\text{INF}_t \end{bmatrix} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \begin{bmatrix} \Delta\text{UNEMP}_{t-1} \\ \text{FF}_{t-1} \\ \Delta\text{INF}_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}.$$

	$\Delta \ln \text{UNEMP}_{t-1}$	FF_{t-1}	ΔINF_{t-1}
$\Delta \ln \text{UNEMP}_t$	0.624 (0.000)	0.015 (0.001)	0.016 (0.267)
FF_t	−0.816 (0.000)	0.979 (0.000)	−0.045 (0.317)
ΔINF_t	−0.501 (0.010)	−0.009 (0.626)	−0.401 (0.000)

Interpreting Estimates

- Variable scale affects cross-parameter estimates
  - Not an issue in ARMA analysis
- Standardizing data can improve interpretation when scales differ

	$\Delta \ln \text{UNEMP}_{t-1}$	FF_{t-1}	ΔINF_{t-1}
$\Delta \ln \text{UNEMP}_t$	0.624 (0.000)	0.015 (0.001)	0.016 (0.267)
FF_t	-0.816 (0.000)	0.979 (0.000)	-0.045 (0.317)
ΔINF_t	-0.501 (0.010)	-0.009 (0.626)	-0.401 (0.000)

- Other important measures – statistical significance, persistence, model selection – are unaffected by standardization

VAR(P) is really a VAR(1)

- Companion form:

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_P \mathbf{y}_{t-P} + \boldsymbol{\epsilon}_t$$

- Reform into a single VAR(1) where $\boldsymbol{\mu} = \mathrm{E}\left[\mathbf{y}_t\right] = \left(\mathbf{I} - \mathbf{\Phi}_1 - \ldots - \mathbf{\Phi}_P\right)^{-1} \mathbf{\Phi}_0$

$$\mathbf{z}_t = \mathbf{\Upsilon} \mathbf{z}_{t-1} + \boldsymbol{\xi}_t$$

$$\mathbf{z}_t = \begin{bmatrix} \mathbf{y}_t - \boldsymbol{\mu} \\ \mathbf{y}_{t-1} - \boldsymbol{\mu} \\ \vdots \\ \mathbf{y}_{t-P+1} - \boldsymbol{\mu} \end{bmatrix}, \quad \mathbf{\Upsilon} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \mathbf{\Phi}_3 & \ldots & \mathbf{\Phi}_{P-1} & \mathbf{\Phi}_P \\ \mathbf{I}_k & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I}_k & \mathbf{0} \end{bmatrix}$$

  - All results can be directly applied to the companion form.
  - Can also be used to transform AR(P) into VAR(1)

Forecasting

Revisiting Univariate Forecasting

- Consider standard AR(1)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$

- Optimal 1-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[y_{t+1}] &= \mathrm{E}_t[\phi_0] + \mathrm{E}_t[\phi_1 y_t] + \mathrm{E}_t[\epsilon_{t+1}] \\ &= \phi_0 + \phi_1 y_t + 0 \end{aligned}$$

- Optimal 2-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[y_{t+2}] &= \mathrm{E}_t[\phi_0] + \mathrm{E}_t[\phi_1 y_{t+1}] + \mathrm{E}_t[\epsilon_{t+2}] \\ &= \phi_0 + \phi_1 \mathrm{E}_t[y_{t+1}] + 0 \\ &= \phi_0 + \phi_1(\phi_0 + \phi_1 y_t) \\ &= \phi_0 + \phi_1\phi_0 + \phi_1^2 y_t \end{aligned}$$

- Optimal h-step ahead forecast:

$$\mathrm{E}_t[y_{t+h}] = \sum_{i=0}^{h-1} \phi_1^i \phi_0 + \phi_1^h y_t$$

Forecasting with VARs

- Identical to univariate case

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

- Optimal 1-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[\mathbf{y}_{t+1}] &= \mathrm{E}_t[\mathbf{\Phi}_0] + \mathrm{E}_t[\mathbf{\Phi}_1 \mathbf{y}_t] + \mathrm{E}_t[\boldsymbol{\epsilon}_{t+1}] \\ &= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_t + \mathbf{0} \end{aligned}$$

- Optimal h-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[y_{t+h}] &= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{\Phi}_0 + \ldots + \mathbf{\Phi}_1^{h-1} \mathbf{\Phi}_0 + \mathbf{\Phi}_1^h \mathbf{y}_t \\ &= \sum_{i=0}^{h-1} \mathbf{\Phi}_1^i \mathbf{\Phi}_0 + \mathbf{\Phi}_1^h \mathbf{y}_t \end{aligned}$$

- Higher order forecast can be recursively computed

$$\mathrm{E}_t[\mathbf{y}_{t+h}] \quad = \quad \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathrm{E}_t[\mathbf{y}_{t+h-1}] + \ldots + \mathbf{\Phi}_P \mathrm{E}_t[\mathbf{y}_{t+h-P}]$$

What makes a good forecast?

- Forecast residuals

$$\hat{e}_{t+h|t} = y_{t+h} - \hat{y}_{t+h|t}$$

- Residuals are *not* white noise
- Can contain an MA($h-1$) component
  - Forecast error for $y_{t+1} - \hat{y}_{t+1|t-h+1}$ was not known at time t.
- Plot your residuals
- Residual ACF
- Mincer-Zarnowitz regressions
- Three period procedure
  - Training sample: Used to build model
  - Validation sample: Used to refine model
  - Evaluation sample: Ultimate test, ideally 1 shot

Multi-step Forecasting

- Two methods
- Iterative method
  - Build model for 1-step ahead forecasts

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

  - Iterate forecast out to period h

$$\hat{\mathbf{y}}_{t+h|t} = \sum_{i=0}^{h-1} \mathbf{\Phi}_1^i \mathbf{\Phi}_0 + \mathbf{\Phi}_1^h \mathbf{y}_t$$

  - Makes efficient use of information
  - Imposes a lot of structure on the problem
- Direct Method
  - Build model for h-step ahead forecasts

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_h \mathbf{y}_{t-h} + \boldsymbol{\epsilon}_t$$

  - Directly forecast using a pseudo 1-step ahead method

$$\hat{\mathbf{y}}_{t+h|t} = \mathbf{\Phi}_0 + \mathbf{\Phi}_h \mathbf{y}_t$$

  - Robust to some nonlinearities

Multi-step Forecast Evaluation

- Multistep forecast evaluation is identical to one-step ahead forecast evaluation with one caveat
- h-step ahead forecast errors may be correlated with any forecast error not known at time t

$$\hat{e}_{t+1|t-h+1}, \hat{e}_{t+2|t-h+2}, \ldots, \hat{e}_{t+h-1|t-1}$$

- Leads to a MA($h-1$) structure in the forecast errors
- Solutions:
  - Use regular GMZ regression with a Newey-West covariance estimator

$$y_{t+h} - \hat{y}_{t+h|t} = \beta_1 + \beta_2 \hat{y}_{t+h|t} + \boldsymbol{\gamma}\mathbf{x}_t + \eta_t$$

$H_0 : \beta_1 = \beta_2 = \boldsymbol{\gamma} = 0$, $H_1 : \beta_1 \neq 0 \cup \beta_2 \neq 0 \cup \gamma_j \neq 0 \; \exists j$

  - Explicitly model the MA($h-1$) and use a standard covariance estimator

$$y_{t+h} - \hat{y}_{t+h|t} = \beta_1 + \beta_2 \hat{y}_{t+h|t} + \boldsymbol{\gamma}\mathbf{x}_t + \eta_t + \sum_{i=1}^{h-1} \theta_i \eta_{t-i}$$

Note: Null is the same; does not impose a restriction on $\boldsymbol{\theta}$

Example: Monetary Policy VAR

- Forecasts produced iteratively for 1 to 8 quarters ahead
- Random walk (FF) or constant mean benchmark
- AR and VAR select lag length using BIC
- Restricted force reversion to in-sample mean using 2-step estimator
  1. Estimate sample mean, and subtract to produce $\tilde{\mathbf{y}}_t = \mathbf{y}_t - \hat{\boldsymbol{\mu}}$
  2. Estimate VAR *without* a constant

$$\tilde{\mathbf{y}}_t = \boldsymbol{\Phi}_1 \tilde{\mathbf{y}}_{t-1} + \ldots + \boldsymbol{\Phi}_P \tilde{\mathbf{y}}_{t-P} + \boldsymbol{\epsilon}_t$$

  3. Forecast and then add the in-sample mean

$$\mathrm{E}_t \left[\tilde{\mathbf{y}}_{t+h}\right] + \hat{\boldsymbol{\mu}}$$

- Evaluation based on relative MSE

$$\text{Rel. MSE} = \frac{\text{MSE}}{\text{MSE}_{bm}}, \ \text{MSE} = {}^{1}\!/_{T-h-R} \sum_{t=R}^{T-h} \left(y_{t+h} - \hat{y}_{t+h|t}\right)^2$$

Example: Monetary Policy VAR

Horizon	Series	VAR Restricted	VAR Unrestricted	AR Restricted	AR Unrestricted
1	Unemployment	0.522	0.520	**0.507**	0.507
	Fed. Funds Rate	**0.887**	0.903	0.923	0.933
	Inflation	0.869	0.868	**0.839**	0.840
2	Unemployment	0.716	**0.710**	0.717	0.718
	Fed. Funds Rate	**0.923**	0.943	*1.112*	*1.130*
	Inflation	*1.082*	*1.081*	*1.031*	*1.030*
4	Unemployment	0.872	**0.861**	0.937	0.940
	Fed. Funds Rate	**0.952**	0.976	*1.082*	*1.109*
	Inflation	*1.000*	0.999	0.998	**0.998**
8	Unemployment	0.820	**0.806**	0.973	0.979
	Fed. Funds Rate	**0.974**	*1.007*	*1.062*	*1.110*
	Inflation	*1.001*	1.000	0.998	**0.997**

Estimation

Estimation and Identification

- Univariate Identification: Box-Jenkins
  - Use ACF and PACF to determine AR and MA lag order
  - Examine residuals
  - Parsimony principle
- The autocorrelation of a scalar process is defined

$$\rho_s = \frac{\gamma_s}{\gamma_0}$$

  where γ_s is s$^{\text{th}}$ the autocovariance
  - Regression coefficient:

$$y_t = \mu + \rho_s y_{t-s} + \epsilon_t$$

- Partial autocorrelation ψ_s
  - Regression interpretation of s$^{\text{th}}$ partial autocorrelation:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_{s-1} y_{t-s+1} + \psi_s y_{t-s} + \epsilon_t$$

  - ψ is the s$^{\text{th}}$ partial autocorrelation

CCF and Partial CCF

- Multivariate equivalents
  - ACF and PACF have same regression definitions
  - Cross-correlation function

$$\rho_{xy,s} = \frac{\mathrm{E}[(x_t - \mu_x)(y_{t-s} - \mu_y)]}{\sqrt{\mathrm{V}[x_t]\mathrm{V}[y_t]}}$$

$$\rho_{yx,s} = \frac{\mathrm{E}[(y_t - \mu_y)(x_{t-s} - \mu_x)]}{\sqrt{\mathrm{V}[x_t]\mathrm{V}[y_t]}}$$

  - Generally different
  - Cross-partial-correlation function $\psi_{xy,s}$

$$\begin{aligned} x_t = & \phi_0 + \phi_{x1} x_{t-1} + \ldots + \phi_{xs-1} x_{t-(s-1)} \\ & + \phi_{y1} y_{t-1} + \ldots + \phi_{ys-1} y_{t-(s-1)} + \varphi_{xy,s} y_{t-s} + \epsilon_{x,t} \end{aligned}$$

    - Can help identify VAR order
- Deeper issue: too many and too complicated
- Simple solution: Model selection

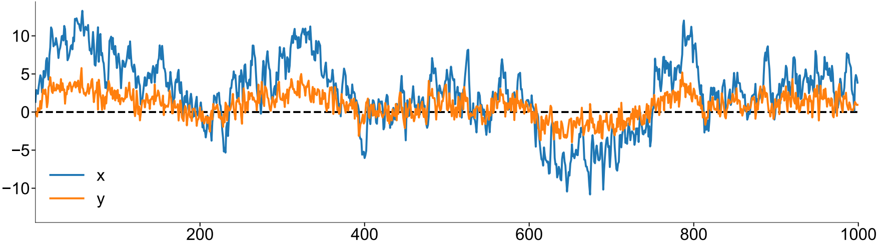
Interpreting CCFs and PCCFs

- y has HAR dynamics, spills over to x

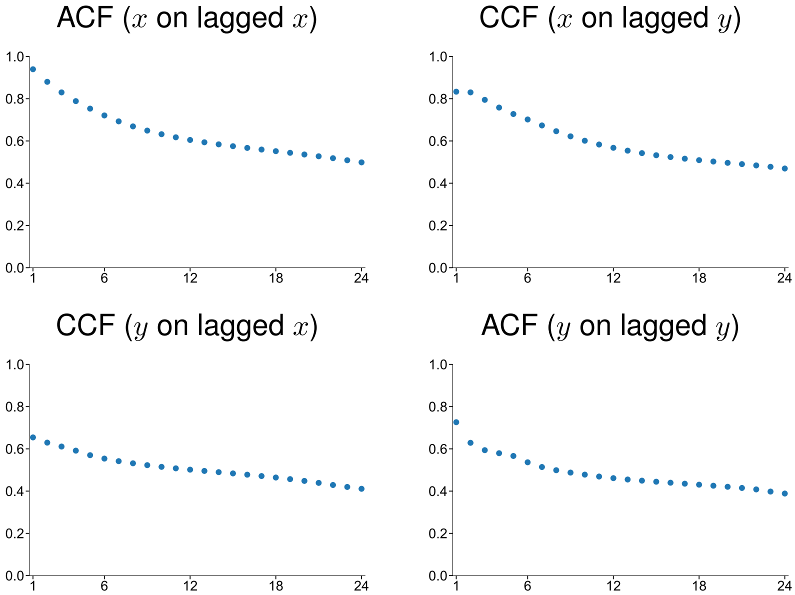
$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0.9 \\ .0 & 0.47 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \sum_{i=2}^{5} \begin{bmatrix} 0 & 0 \\ 0 & 0.06 \end{bmatrix} \begin{bmatrix} x_{t-i} \\ y_{t-i} \end{bmatrix}$$

$$+ \sum_{j=6}^{22} \begin{bmatrix} 0 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} x_{t-j} \\ y_{t-j} \end{bmatrix} + \begin{bmatrix} \epsilon_{x,t} \\ \epsilon_{y,t} \end{bmatrix}$$

- Simulated data

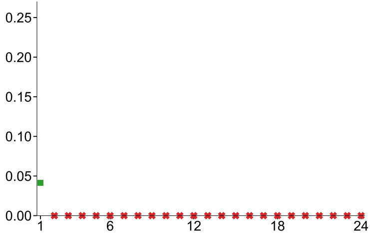


ACFs and CCFs

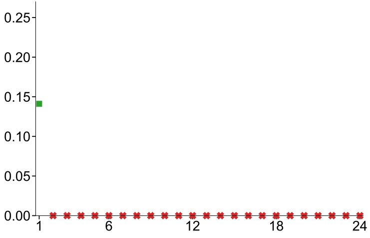


PACFs and Partial CCFs

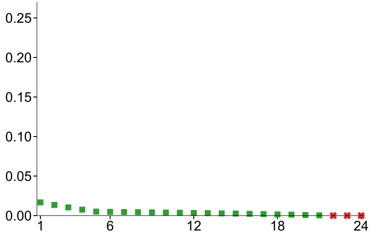
PACF (x on lagged x)



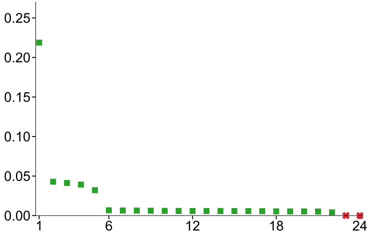
PCCF (x on lagged y)



PCCF (y on lagged x)



PACF (y on lagged y)



Model Selection

- Step 1: Pick maximum lag length
  - Information criteria

$$\text{AIC:} \qquad \ln|\mathbf{\Sigma}(P)| + k^2 P \frac{2}{T}$$

$$\text{Hannan-Quinn IC (HQIC):} \qquad \ln|\mathbf{\Sigma}(P)| + k^2 P \frac{\ln \ln T}{T}$$

$$\text{SIC:} \qquad \ln|\mathbf{\Sigma}(P)| + k^2 P \frac{\ln T}{T}$$

    - $\Sigma(P)$ is the covariance of the residuals using P lags
    - $|\cdot|$ is the determinant
  - Hypothesis testing based
    - General to Specific
    - Specific to General
  - Likelihood Ratio

$$\left(T - P_2 k^2\right)\left(\ln|\mathbf{\Sigma}(P_1)| - \ln|\mathbf{\Sigma}(P_2)|\right) \overset{A}{\sim} \chi^2_{(P_2-P_1)k^2}$$

Lag Length Selection in Monetary Policy VAR

- Maximum lag: 12 (1 year)

Lag Length	AIC	HQIC	BIC	LR	P-val
0	4.014	3.762	3.605	925	0.000
1	0.279	0.079	0.000▼▲	39.6	0.000
2	0.190	0.042	0.041	40.9	0.000
3	0.096	0.000▼	0.076	29.0	0.001
4	0.050▼	0.007	0.160	7.34	0.602▼
5	0.094	0.103	0.333	29.5	0.001
6	0.047	0.108	0.415	13.2	0.155
7	0.067	0.180	0.564	32.4	0.000
8	0.007	0.172▲	0.634	19.8	0.019
9	0.000▲	0.217	0.756	7.68	0.566▲
10	0.042	0.312	0.928	13.5	0.141
11	0.061	0.382	1.076	13.5	0.141
12	0.079	0.453	1.224	–	–

Granger Causality

Granger Causality

- First fundamentally new concept
- Examines whether lags of one variable are helpful in predicting another

Definition (Granger Causality)

A scalar random variable $\{x_t\}$ is said to **not** Granger cause $\{y_t\}$ if $\mathrm{E}[y_t|x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}, \ldots] = \mathrm{E}[y_t|, y_{t-1}, y_{t-2}, \ldots]$. That is, $\{x_t\}$ does not Granger cause if the forecast of y_t is the same whether conditioned on past values of x_t or not.

Granger Causality

- Translates directly into a restriction in a VAR
- Unrestricted

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Restricted so that x_t does not GC y_t

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$x_t = \phi_{01} + \phi_{11} x_{t-1} + \phi_{12} y_{t-1} + \epsilon_{1,t}$$

$$y_t = \phi_{02} + \phi_{22} y_{t-1} + \epsilon_{2,t} \Leftarrow \text{ No } x_t!$$

More Granger Causality

- In P lag model

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_P \mathbf{y}_{t-P} + \boldsymbol{\epsilon}_t$$

  the null hypothesis is

$$H_0 : \phi_{ij,1} = \phi_{ij,2} = \ldots = \phi_{ij,P} = 0$$

- Alternative is

$$H_0 : \phi_{ij,1} \neq 0 \text{ or } \phi_{ij,2} \neq 0 \text{ or } \ldots \text{ or } \phi_{ij,P} \neq 0$$

- Likelihood Ratio test

$$\left(T - Pk^2\right)\left(\ln|\mathbf{\Sigma}_r| - \ln|\mathbf{\Sigma}_u|\right) \stackrel{A}{\sim} \chi^2_P$$

- $\mathbf{\Sigma}_u$ is the covariance of the errors from unrestricted model
- $\mathbf{\Sigma}_r$ is the covariance of the errors from restricted model
- $T - Pk^2$ is number of observations minus number of free parameters in unrestricted model
  - Why χ^2_P?

Monetary Policy VAR

- Standard tool in monetary policy analysis
  - Unemployment rate (differenced)
    - Federal Funds rate
    - Inflation rate (differenced)

$$\begin{bmatrix} \Delta \text{UNEMP}_t \\ \text{FF}_t \\ \Delta \text{INF}_t \end{bmatrix} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \begin{bmatrix} \Delta \text{UNEMP}_{t-1} \\ \text{FF}_{t-1} \\ \Delta \text{INF}_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}.$$

Granger Causality in Campbell's VAR

- Using model with lags 3 lags (HQIC)
- $H_0 : \phi_{ij,1} = \phi_{ij,2} = \phi_{ij,3} = 0$
- $H_1 : \phi_{ij,1} \neq 0$ or $\phi_{ij,2} \neq 0$ or $\phi_{ij,3} \neq 0$
- i represent series being affected by lags of series j

	Fed. Funds Rate		Inflation		Unemployment	
Exclusion	P-val	Stat	P-val	Stat	P-val	Stat
Fed. Funds Rate	–	–	0.001	13.068	0.014	8.560
Inflation	0.001	14.756	–	–	0.375	1.963
Unemployment	0.000	19.586	0.775	0.509	–	–
All	0.000	33.139	0.000	18.630	0.005	10.472

Impulse Response Functions

Impulse Response Functions

- Second fundamentally new concept
- Complicated dynamics of a VAR make direct interpretation of coefficients difficult
- Solution is to examine impulse responses
- The impulse response function of y_i with respect to a shock in ϵ_j, for any j and i, is defined as the change in y_{it+s}, $s \geq 0$ for a unit shock in ϵ_{jt}
  - Hard to decipher
- As long as $\mathbf{y}_t$ is covariance stationarity it must have a VMA representation,

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\epsilon}_t + \boldsymbol{\Xi}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\Xi}_2 \boldsymbol{\epsilon}_{t-2} + \ldots$$

- $\boldsymbol{\Xi}_j$ are the impulse responses!
- Why?
  - Directly measure the effect in period j of any shock

AR(P) and MA(∞)

- Any stationary AR(P)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_P y_{t-P} + \epsilon_t$$

can be represented as an MA(∞)

$$y_t = \phi_0/(1 - \phi_1 - \phi_2 - \ldots - \phi_P) + \epsilon_t + \sum_{i=1}^{\infty} \theta_i \epsilon_{t-i}$$

- AR(1)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$

becomes

$$y_t = \phi_0/(1 - \phi_1) + \epsilon_t + \sum_{i=1}^{\infty} \phi_1^i \epsilon_{t-i}$$

- Stationary VAR(P) have the same relationship to VMA(∞)

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_P \mathbf{y}_{t-P} + \boldsymbol{\epsilon}_t$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\epsilon}_t + \mathbf{\Xi}_1 \boldsymbol{\epsilon}_{t-1} + \mathbf{\Xi}_2 \boldsymbol{\epsilon}_{t-2} + \ldots$$

- Easy in VAR(1)

$$\mathbf{y}_t = (\mathbf{I}_K - \mathbf{\Phi}_1)^{-1}\mathbf{\Phi}_0 + \boldsymbol{\epsilon}_t + \mathbf{\Phi}_1\boldsymbol{\epsilon}_{t-1} + \mathbf{\Phi}_1^2\boldsymbol{\epsilon}_{t-2} + \ldots.$$

- $\mathbf{\Xi}_j = \mathbf{\Phi}_1^j$
- In the general VAR(P),

$$\mathbf{\Xi}_j = \mathbf{\Phi}_1\mathbf{\Xi}_{j-1} + \mathbf{\Phi}_2\mathbf{\Xi}_{j-2} + \ldots + \mathbf{\Phi}_P\mathbf{\Xi}_{j-P}$$

  where $\mathbf{\Xi}_0 = \mathbf{I}_k$ and $\mathbf{\Xi}_m = \mathbf{0}$ for $m < 0$.
  - In a VAR(2),

$$\mathbf{y}_t = \mathbf{\Phi}_1\mathbf{y}_{t-1} + \mathbf{\Phi}_2\mathbf{y}_{t-2} + \boldsymbol{\epsilon}_t$$

    - $\mathbf{\Xi}_0 = \mathbf{I}_k$, $\mathbf{\Xi}_1 = \mathbf{\Phi}_1$, $\mathbf{\Xi}_2 = \mathbf{\Phi}_1^2 + \mathbf{\Phi}_2$, and $\mathbf{\Xi}_3 = \mathbf{\Phi}_1^3 + \mathbf{\Phi}_1\mathbf{\Phi}_2 + \mathbf{\Phi}_2\mathbf{\Phi}_1$.
- Confidence intervals are also somewhat painful
  - Explained in notes

Considerations for Shocks

- Simple bivariate VAR(1)

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- How you *shock* matters
- Depends on correlation between $\epsilon_{1,t}$ and $\epsilon_{2,t}$
- 3 methods
  - Ignore correlation and just shock $\epsilon_{j,t}$ with a 1 standard deviation shock
  - Use Cholesky to factor $\mathbf{\Sigma}$ and use $\mathbf{\Sigma}^{1/2}\mathbf{e}_j$ where $\mathbf{e}_j$ is a vector of zeros with 1 in the j$^{\text{th}}$ position

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix} \quad \mathbf{\Sigma}_C^{1/2} = \begin{bmatrix} 1 & 0 \\ .5 & .866 \end{bmatrix}$$

    - Variable order matters
  - "Generalized" impulse response that uses a projection method

Example of the different shocks

- Define the error covariance

$$\mathbf{\Sigma} = \left[\begin{array}{cc} \sigma_x^2 & \sigma_x \sigma_y \rho \\ \sigma_x \sigma_y \rho & \sigma_y^2 \end{array} \right]$$

  - Standardized

$$\left[\begin{array}{c} 0 \\ \sigma_x \end{array} \right]$$

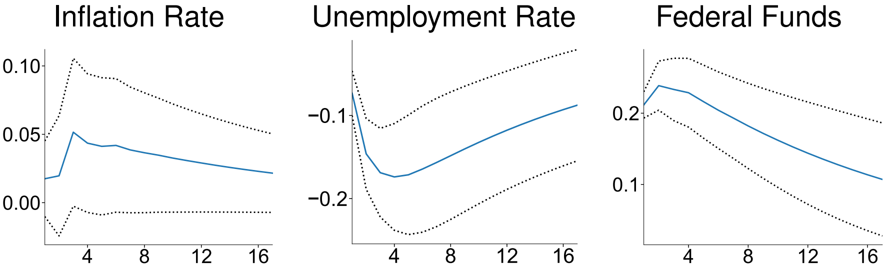
  - Cholesky

$$\mathbf{\Sigma}_C^{1/2} \left[\begin{array}{c} 0 \\ 1 \end{array} \right] = \left[\begin{array}{cc} \sigma_x & 0 \\ \sigma_y \rho & \sigma_y \sqrt{1-\rho^2} \end{array} \right] \left[\begin{array}{c} 0 \\ 1 \end{array} \right] = \left[\begin{array}{c} 0 \\ \sigma_y \sqrt{1-\rho^2} \end{array} \right]$$

$$\left[\begin{array}{cc} \sigma_x & 0 \\ \sigma_y \rho & \sigma_y \sqrt{1-\rho^2} \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = \left[\begin{array}{c} \sigma_x \\ \sigma_y \rho \end{array} \right]$$

Impulse Responses

- Federal Funds ordered first
- Response to Federal Funds Shock
- Cholesky factorization



$$W_t = \Phi W_{t-1} + u_t$$

$(z^2 - \phi_1 z - \phi_2)$ $\qquad c_1 = \lambda_1$

$(z - c_1)(z - c_2)$ $\qquad c_2 = \lambda_2$