Financial Econometrics
HT Week 1 Assignment Answers

January 31, 2019

Exercise 7.2

Derive explicit relationships between the parameters of an APARCH(1,1,1) and

i. **ARCH(1)**
The APARCH is

$$\sigma_t^\delta = \omega + \alpha\left(|\epsilon_{t-1}| + \gamma\epsilon_{t-1}\right)^\delta + \beta\sigma_{t-1}^\delta$$

so when $\gamma = \beta = 0$ and $\delta = 2$ then

$$\sigma_t^2 = \omega + \alpha\left|\epsilon_{t-1}\right|^2$$

ii. **GARCH(1,1)**
$\gamma = 0$ and $\delta = 2$, so that

$$\sigma_t^2 = \omega + \alpha\left|\epsilon_{t-1}\right|^2 + \beta\sigma_{t-1}^2$$

iii. **AVGARCH(1,1)**

$$\sigma_t = \omega + \alpha\left|\epsilon_{t-1}\right| + \beta\sigma_{t-1}$$

iv. **TARCH(1,1,1)**

$$\sigma_t^\delta = \omega + \alpha\left(|\epsilon_{t-1}| + \gamma\epsilon_{t-1}\right)^\delta + \beta\sigma_{t-1}^\delta$$

which is the same as

$$\sigma_t = \omega + \tilde{\alpha}\left|\epsilon_{t-1}\right| + \tilde{\gamma}\left|\epsilon_{t-1}\right| I_{[\epsilon_{t-1}<0]} + \beta\sigma_{t-1}$$

with $\tilde{\alpha} = \alpha + \alpha\gamma$ and $\tilde{\alpha} + \tilde{\gamma} = \alpha - \alpha\gamma$.

v. **GJR-GARCH(1,1,1)**

$$\begin{aligned}\sigma_t^2 &= \omega + \alpha\left(|\epsilon_{t-1}| + \gamma\epsilon_{t-1}\right)^2 + \beta\sigma_{t-1}^2 \\ &= \omega + \left(\alpha + \gamma^2\right)\epsilon_{t-1}^2 + 2\gamma\epsilon_{t-1} + \beta\sigma_{t-1}^2\end{aligned}$$

which is the same as

$$\sigma_t = \omega + \tilde{\alpha}\epsilon_{t-1}^2 + \tilde{\gamma}\epsilon_{t-1}^2 I_{[\epsilon_{t-1}<0]} + \beta\sigma_{t-1}^2$$

so that when $\tilde{\alpha} = \alpha + \gamma^2 + 2\gamma$ and $\tilde{\alpha} + \tilde{\gamma} = \alpha + \gamma^2 - 2\gamma$.

Exercise 7.4

Let r_t follow a GARCH process

$$
\begin{aligned}
r_t &= \sigma_t e_t \\
\sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 \\
e_t &\overset{\text{i.i.d.}}{\sim} N(0,1)
\end{aligned}
$$

1. **What are the values of the following quantities?**

   (a) $\mathrm{E}[r_{t+1}] = \mathrm{E}[e_{t+1}\sigma_{t+1}] = \mathrm{E}[\mathrm{E}_t[e_{t+1}\sigma_{t+1}]] = \mathrm{E}[\mathrm{E}_t[e_{t+1}]\sigma_{t+1}] = \mathrm{E}[0\sigma_{t+1}] = 0$

   (b) $\mathrm{E}_t[r_{t+1}] = \mathrm{E}_t[e_{t+1}\sigma_{t+1}] = \mathrm{E}_t[e_{t+1}]\sigma_{t+1} = 0\sigma_{t+1} = 0$

   (c) $\mathrm{V}[r_{t+1}] = \mathrm{E}[r_{t+1}^2] = \mathrm{E}[e_{t+1}^2\sigma_{t+1}^2] = \mathrm{E}\left[\mathrm{E}_t\left[e_{t+1}^2\sigma_{t+1}^2\right]\right] = \mathrm{E}[1 \times \sigma_{t+1}^2] = \bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$

   (d) $\mathrm{V}_t[r_{t+1}] = \mathrm{E}_t[e_{t+1}^2\sigma_{t+1}^2] = \mathrm{E}_t[e_{t+1}^2]\sigma_{t+1}^2 = 1 \times \sigma_{t+1}^2 = \sigma_{t+1}^2$

   (e) ρ_1

$$
\begin{aligned}
\rho_1 &= \frac{\mathrm{E}[(e_t\sigma_t)(e_{t-1}\sigma_{t-1})]}{\mathrm{V}[r_t]} \\
&= \frac{\mathrm{E}[\mathrm{E}_{t-1}[e_t e_{t-1}\sigma_t\sigma_{t-1}]]}{\mathrm{V}[r_t]} \\
&= \frac{\mathrm{E}[\mathrm{E}_{t-1}[e_t]e_{t-1}\sigma_t\sigma_{t-1}]}{\mathrm{V}[r_t]} \\
&= \frac{\mathrm{E}[0e_{t-1}\sigma_t\sigma_{t-1}]}{\mathrm{V}[r_t]} \\
&= 0
\end{aligned}
$$

2. **What is** $\mathrm{E}[(r_t^2 - \bar{\sigma}^2)(r_{t-1}^2 - \bar{\sigma}^2)]$

   The ACF of an GARCH(1,1) can be derived by mapping it into an ARMA(1,1) by adding $\left(r_t^2 - \sigma_t^2\right)$ to both sides (or you can add and subtract r_t^2 from the left side and then move the term $-r_t^2 + \sigma_t^2$ to the right-hand side), and then adding and subtracting βr_{t-1}^2 on the right hand side.

$$
\begin{aligned}
\sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2 \\
\sigma_t^2 + \left(r_t^2 - \sigma_t^2\right) &= \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + \alpha r_{t-1}^2 + \beta r_{t-1}^2 - \beta r_{t-1}^2 + \beta\sigma_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + (\alpha+\beta) r_{t-1}^2 - \beta\left(r_{t-1}^2 - \sigma_{t-1}^2\right) + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + (\alpha+\beta) r_{t-1}^2 - \beta\nu_{t-1} + \nu_t
\end{aligned}
$$

   From here we can apply the formula in the time-series notes to get the first autocovariance, which is

$$
\frac{\mathrm{V}[\nu_t]\,\alpha\,(1-\beta\,(\alpha+\beta))}{1-(\alpha+\beta)^2}
$$

3. **Describe the h-step ahead forecast from this model.**

$$\begin{aligned}
\mathrm{E}_t[\sigma^2_{t+1}] &= \mathrm{E}_t[\omega + \alpha r_t^2 + \beta\sigma_t^2] \\
&= \omega + \alpha r_t^2 + \beta\sigma_t^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}_t[\sigma^2_{t+2}] &= \mathrm{E}_t[\omega + \alpha r_{t+1}^2 + \beta\sigma_{t+1}^2] \\
&= \omega + \alpha\mathrm{E}_t[r_{t+1}^2] + \beta\mathrm{E}_t[\sigma_{t+1}^2] \\
&= \omega + \alpha\mathrm{E}_t[e_{t+1}^2\sigma_{t+1}^2] + \beta\mathrm{E}_t[\sigma_{t+1}^2] \\
&= \omega + \alpha\mathrm{E}_t[e_{t+1}^2]\sigma_{t+1}^2 + \beta\sigma_{t+1}^2 \\
&= \omega + \alpha\sigma_{t+1}^2 + \beta\sigma_{t+1}^2 \\
&= \omega + (\alpha+\beta)\,\sigma_{t+1}^2
\end{aligned}$$

and substituting σ^2_{t+1}, which is known at time t, will produce

$$\begin{aligned}
\mathrm{E}_t[\sigma^2_{t+2}] &= \omega + (\alpha+\beta)\left(\omega + \alpha r_t^2 + \beta\sigma_t^2\right) \\
&= \omega + (\alpha+\beta)\,\omega + (\alpha+\beta)\left(\alpha r_t^2 + \beta\sigma_t^2\right)
\end{aligned}$$

Finally note that $\mathrm{E}_t[\sigma^2_{t+3}] = \omega + \alpha\mathrm{E}_t[r^2_{t+2}] + \beta\mathrm{E}_t[\sigma^2_{t+2}]$, and so

$$\mathrm{E}_t[\sigma^2_{t+3}] = \omega + (\alpha+\beta)\,\omega + (\alpha+\beta)^2\,\omega + (\alpha+\beta)^2\left(\alpha r_t^2 + \beta\sigma_t^2\right)$$

and the pattern emerges,

$$\mathrm{E}_t[\sigma^2_{t+h}] = \sum_{i=0}^{h-1}(\alpha+\beta)^i\,\omega + (\alpha+\beta)^{h-1}\left(\alpha r_t^2 + \beta\sigma_t^2\right)$$

The h-step ahead forecast is an exponentially declining function of the time $t+1$ forecast plus a constant. For large h, the forecast converges to $\bar{\sigma}^2$.