Financial Econometrics
HT Week 6 Assignment Answers

March 7, 2019

Exercise 5.4

Data on interest rates on US government debt was collected for 3-month ($3MO$) T-bills, and 3-year ($3YR$) and 10-year ($10YR$) bonds from 1957 until 2009. Using these three series, the following variables were defined

Level	$3MO$
Slope	$10YR - 3MO$
Curvature	$(10YR - 3YR) - (3YR - 3MO)$

1. **In terms of VAR analysis, does it matter whether the original data or the level-slope-curvature model is fit? Hint: Think about reparameterizations between the two.** No, the two parameterizations are equivalent since one is just a rotation of the other. In other words, the original data (x_t) and the LSC data (y_t) are related by

$$y_t = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} x_t$$

where the rotation matrix (which adds and subtracts) is clearly full rank. VAR estimates are ultimately a series of OLS estimates which are known to be invariant to this type of transformation. It will be the case that the R^2 and parameter inference will be unaffected by this change.

Granger Causality analysis was performed on this set and the p-vals were

	Level_{t-1}	Slope_{t-1}	Curvature_{t-1}
Level_t	0.000	0.244	0.000
Slope_t	0.000	0.000	0.000
Curvature_t	0.000	0.000	0.000
All (excl. self)	0.000	0.000	0.000

2. **Interpret this table.**

   The table in the notes has a problem where the bottom row should have been the final column. This is fixed here. The table contains results from Granger Causality tests which test that variable j can be excluded from the model for variable i. GC tests are standard linear restriction tests and can always be written as $H_0 : \phi_{ij,1} = \phi_{ij,2} = \phi_{ij,P}$ where P is the VAR order. The alternative is that the any of the parameters are not zero. GC is not a test of causality in the usual sense but is more of a test of whether a forecasting model for series i should include series j.

3. **When constructing impulse response graphs the selection of the covariance of the shocks is important. Outline the alternatives and describe situations when each may be preferable.**

   There are 4 choices:

   - Uncorrelated - Under this assumption a shock to series i has an impact in period 0 only to series i and so only spills over into series j in the future through the VAR coefficients. It is the most restrictive and is probably not appropriate in most situations. On the other hand, the shocks are easy to identify to belonging to series i.
   - Choleski - The Choleski orders shocks so that a shock to series i spills over to series j as long as $j \geq i$ in period 0. for series where $j < i$ the shock can only spillover in subsequent period. Using this type of covariance matrix is making an assumption about the speed of the reaction to the different series. If one series is fast (e.g. a financial asset) and another is slow (e.g. a real macroeconomic variable) then it may be the case that the shock to the slow variable spills over immediately into the fast variable, but that a shock to the fast variable does not immediately spill over into the slow variable. This type of ordering is useful if the reaction time of the variable can be ordered.
   - Generalized - The Generalized IR is a repeated application of the Choleski IR where the effect of a shock to series i is captured by reordering the VAR so that series i is the first series for $i = 1, 2, \ldots, k$. Reordering means that the order of the variables in the original VAR is not important. It also means that all shocks affect all series in period 0 so it may be difficult to tell what exactly is a shock.
   - Spectral (Symmetric) - The symmetric square root assumes all spillover into all in period 0. It is similar to the Generalized in that the order does not matter and that identifying a shock to belong to series i is difficult.

4. **The figure contains the impulse response curves for this model. Interpret the graph. Also comment on why the impulse responses can all be significantly different from 0 in light of the Granger Causality table.**

   A shock to level has a strong, positive, persistent effect on the future level. The level shock also has an immediate negative effect on the slope which is reasonable since the slope has the level appear with a negative sign. The level shock as a short run increast in the cuvature but a logn run decrease, which is statistically positive. The short run increase is due to the positive sign of level in the equation for curvature. The long-run negative effect is driven by the VAR parameters.

   Slope shocks lead to increases in slope, although with less persistence that the level shocks on level. Increases in the slope lead to less curvature, which makes sense since a slope change means the 10 year rate has risen by the 3 month has not, and to the line will appear "flatter". Slope shocks eventually feedback into level shocks which is consistent with mean reversion in the average slope of the yield curve.

Curvature shocks are also persistent. A shock to the curvature leads, on average to an increase in the slope and a decrease in the level. These two are internally consistent since a decrease in the level should, all things equal, increase the slope.

5. **Why are some of the "0" lag impulses 0 while other aren't?**

   This graph appears to be using a Choleski covariance square root since level affects all in period 0, slope does not affect level in period 0, and curvature does not affect the others in period 2.

Exercise 5.5

1. **Consider the AR(2)**

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$$

   (a) **Rewrite the model with Δy_t on the left-hand side and y_{t-1} and Δy_{t-1} on the right-hand side.**

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t \\ y_t - y_{t-1} &= \phi_1 y_{t-1} - y_{t-1} + \phi_2 y_{t-2} + \epsilon_t \\ y_t - y_{t-1} &= \phi_1 y_{t-1} - y_{t-1} + \phi_2 y_{t-1} - \phi_2 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t \\ \Delta y_t &= (\phi_1 + \phi_2 - 1)\, y_{t-1} - \phi_2 \Delta y_{t-1} + \epsilon_t \end{aligned}$$

   (b) **What restrictions are needed on ϕ_1 and ϕ_2 for this model to collapse to an AR(1) in the first differences?**
   $\phi_1 + \phi_2 = 1$

   (c) **When the model collapses, what does this tell you about y_t?**
   The process contains a unit root since when the coefficients add to 1 this must be the case.

2. **Consider the VAR(1)**

$$\begin{aligned} x_t &= x_{t-1} + \epsilon_{1,t} \\ y_t &= \beta x_{t-1} + \epsilon_{2,t} \end{aligned}$$

   where $\{\epsilon_t\}$ is a vector white noise process.

   (a) **Are x_t and y_t cointegrated?**
   Yes. x_t is obviously I(1) since it is a standard random walk. y_t depends on an I(1) process plus an error, and so $y_t - \beta x_{t-1} = \epsilon_{2,t} \sim I(0)$

   (b) **Write this model in error correction form.**

$$\begin{aligned} \begin{bmatrix} x_t \\ y_t \end{bmatrix} - \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} &= \begin{bmatrix} x_{t-1} \\ \beta x_{t-1} \end{bmatrix} - \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \\ \begin{bmatrix} \Delta x_t \\ \Delta y_t \end{bmatrix} &= \begin{bmatrix} 0 \\ \beta x_{t-1} - y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \\ \begin{bmatrix} \Delta x_t \\ \Delta y_t \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\beta \quad -1] \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \end{aligned}$$

3. **Consider the VAR(1)**

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.625 & -0.3125 \\ -0.75 & 0.375 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

where $\{\epsilon_t\}$ is a vector white noise process.

(a) **Verify that x_t and y_t are cointegrated.**
The simple method is to write as an VECM and to verify πhas rank 1. The VECM is

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.625 & -0.3125 \\ -0.75 & 0.375 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} - \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 0.625 & -0.3125 \\ -0.75 & 0.375 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} - \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_t \\ \Delta y_t \end{bmatrix} = \left[\begin{bmatrix} 0.625 & -0.3125 \\ -0.75 & 0.375 \end{bmatrix} - \mathbf{I}_2\right] \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} -0.375 & -0.3125 \\ -0.75 & -0.625 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

This matrix is obviously rank 1 since the second row is simply -2 times the first. We can also put it into row echelon form,

$$\begin{bmatrix} -0.375 & -0.3125 \\ -0.75 & -0.625 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5/6 \\ -0.75 & -0.625 \end{bmatrix} \textbf{ Divide for 1 by first element}$$

$$\Rightarrow \begin{bmatrix} 1 & 5/6 \\ 0 & 0 \end{bmatrix} \textbf{ Add } 3/4 \textbf{ row 1 to row 2}$$

(b) **Write this model in error correction form.**
See answer to (a).

(c) **Compute the speed of adjustment coefficient $\boldsymbol{\alpha}$ and the cointegrating vector $\boldsymbol{\beta}$ where the $\boldsymbol{\beta}$ on x_t is normalized to 1.**
The row echelon form contains the cointegrating vector, $\boldsymbol{\beta} = [1 \ 5/6]'$ and so $\boldsymbol{\alpha} = [-0.375 \ -0.75]'$ is the first column of the coefficient matrix in the reduced form.