Financial Econometrics
HT Week 3 Assignment Answers

February 14, 2019

Exercise 7.4

Let r_t follow a GARCH process

$$
\begin{aligned}
r_t &= \sigma_t e_t \\
\sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 \\
e_t &\overset{\text{i.i.d.}}{\sim} N(0,1)
\end{aligned}
$$

1. **What are the values of the following quantities?**

   (a) $\mathrm{E}[r_{t+1}] = \mathrm{E}[e_{t+1}\sigma_{t+1}] = \mathrm{E}[\mathrm{E}_t[e_{t+1}\sigma_{t+1}]] = \mathrm{E}[\mathrm{E}_t[e_{t+1}]\sigma_{t+1}] = \mathrm{E}[0\sigma_{t+1}] = 0$

   (b) $\mathrm{E}_t[r_{t+1}] = \mathrm{E}_t[e_{t+1}\sigma_{t+1}] = \mathrm{E}_t[e_{t+1}]\sigma_{t+1} = 0\sigma_{t+1} = 0$

   (c) $\mathrm{V}[r_{t+1}] = \mathrm{E}[r_{t+1}^2] = \mathrm{E}[e_{t+1}^2\sigma_{t+1}^2] = \mathrm{E}\left[\mathrm{E}_t\left[e_{t+1}^2\sigma_{t+1}^2\right]\right] = \mathrm{E}[1\times\sigma_{t+1}^2] = \bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$

   (d) $\mathrm{V}_t[r_{t+1}] = \mathrm{E}_t[e_{t+1}^2\sigma_{t+1}^2] = \mathrm{E}_t[e_{t+1}^2]\sigma_{t+1}^2 = 1\times\sigma_{t+1}^2 = \sigma_{t+1}^2$

   (e) ρ_1

$$
\begin{aligned}
\rho_1 &= \frac{\mathrm{E}[(e_t\sigma_t)(e_{t-1}\sigma_{t-1})]}{\mathrm{V}[r_t]} \\
&= \frac{\mathrm{E}[\mathrm{E}_{t-1}[e_t e_{t-1}\sigma_t\sigma_{t-1}]]}{\mathrm{V}[r_t]} \\
&= \frac{\mathrm{E}[\mathrm{E}_{t-1}[e_t]e_{t-1}\sigma_t\sigma_{t-1}]}{\mathrm{V}[r_t]} \\
&= \frac{\mathrm{E}[0e_{t-1}\sigma_t\sigma_{t-1}]}{\mathrm{V}[r_t]} \\
&= 0
\end{aligned}
$$

2. **What is** $\mathrm{E}[(r_t^2-\bar{\sigma}^2)(r_{t-1}^2-\bar{\sigma}^2)]$

   The ACF of an GARCH(1,1) can be derived by mapping it into an ARMA(1,1) by adding $\left(r_t^2-\sigma_t^2\right)$ to both sides (or you can add and subtract r_t^2 from the left side and then move the term $-r_t^2+\sigma_t^2$ to the right-hand side), and then adding and subtracting βr_{t-1}^2 on the right hand side.

$$
\begin{aligned}
\sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 \\
\sigma_t^2 + \left(r_t^2 - \sigma_t^2\right) &= \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + \alpha r_{t-1}^2 + \beta r_{t-1}^2 - \beta r_{t-1}^2 + \beta \sigma_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + (\alpha + \beta) r_{t-1}^2 - \beta \left(r_{t-1}^2 - \sigma_{t-1}^2\right) + \left(r_t^2 - \sigma_t^2\right) \\
r_t^2 &= \omega + (\alpha + \beta) r_{t-1}^2 - \beta \nu_{t-1} + \nu_t
\end{aligned}
$$

From here we can apply the formula in the time-series notes to get the first autocovariance, which is

$$
\frac{\mathrm{V}[\nu_t]\,\alpha\left(1 - \beta(\alpha + \beta)\right)}{1 - (\alpha + \beta)^2}
$$

3. **Describe the h-step ahead forecast from this model.**

$$
\begin{aligned}
\mathrm{E}_t[\sigma_{t+1}^2] &= \mathrm{E}_t[\omega + \alpha r_t^2 + \beta \sigma_t^2] \\
&= \omega + \alpha r_t^2 + \beta \sigma_t^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}_t[\sigma_{t+2}^2] &= \mathrm{E}_t[\omega + \alpha r_{t+1}^2 + \beta \sigma_{t+1}^2] \\
&= \omega + \alpha \mathrm{E}_t[r_{t+1}^2] + \beta \mathrm{E}_t[\sigma_{t+1}^2] \\
&= \omega + \alpha \mathrm{E}_t[e_{t+1}^2 \sigma_{t+1}^2] + \beta \mathrm{E}_t[\sigma_{t+1}^2] \\
&= \omega + \alpha \mathrm{E}_t[e_{t+1}^2] \sigma_{t+1}^2 + \beta \sigma_{t+1}^2 \\
&= \omega + \alpha \sigma_{t+1}^2 + \beta \sigma_{t+1}^2 \\
&= \omega + (\alpha + \beta)\, \sigma_{t+1}^2
\end{aligned}
$$

and substituting σ_{t+1}^2, which is known at time t, will produce

$$
\begin{aligned}
\mathrm{E}_t[\sigma_{t+2}^2] &= \omega + (\alpha + \beta)\left(\omega + \alpha r_t^2 + \beta \sigma_t^2\right) \\
&= \omega + (\alpha + \beta)\,\omega + (\alpha + \beta)\left(\alpha r_t^2 + \beta \sigma_t^2\right)
\end{aligned}
$$

Finally note that $\mathrm{E}_t[\sigma_{t+3}^2] = \omega + \alpha \mathrm{E}_t[r_{t+2}^2] + \beta \mathrm{E}_t[\sigma_{t+2}^2]$, and so

$$
\mathrm{E}_t[\sigma_{t+3}^2] = \omega + (\alpha + \beta)\,\omega + (\alpha + \beta)^2\,\omega + (\alpha + \beta)^2\left(\alpha r_t^2 + \beta \sigma_t^2\right)
$$

and the pattern emerges,

$$\mathrm{E}_t[\sigma^2_{t+h}] = \sum_{i=0}^{h-1} (\alpha+\beta)^i\,\omega + (\alpha+\beta)^{h-1}\left(\alpha r_t^2 + \beta\sigma_t^2\right)$$

The h-step ahead forecast is an exponentially declining function of the time $t+1$ forecast plus a constant. For large h, the forecast converges to $\bar{\sigma}^2$.

Exercise 7.16

Suppose $\{y_t\}$ is covariance stationary and can be described by the following process:

$$\begin{aligned} y_t &= \phi_0 + \phi_1 y_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t \\ \epsilon_t &= \sigma_t e_t \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 \\ e_t &\overset{\text{i.i.d.}}{\sim} N(0,1) \end{aligned}$$

1. **What are the values of the following quantities:**

(a) $\mathrm{E}[y_{t+1}]$

$$\begin{aligned} \mathrm{E}[y_{t+1}] &= \mathrm{E}[\phi_0 + \phi_1 y_t + \theta_1 \epsilon_t + \epsilon_{t+1}] \\ &= \phi_0 + \phi_1 \mathrm{E}[y_t] + \theta_1 \mathrm{E}[\epsilon_t] + \mathrm{E}[\epsilon_{t+1}] \\ &= \phi_0 + \phi_1 \mathrm{E}[y_t] + 0 + 0 \\ \Rightarrow \mathrm{E}[y_{t+1}] &= \phi_0 + \phi_1 \mathrm{E}[y_t] \\ \Rightarrow \mathrm{E}[y_{t+1}] &= \phi_0/1-\phi_1 \end{aligned}$$

(b) $\mathrm{E}_t[y_{t+1}] = \mathrm{E}_t[\phi_0 + \phi_1 y_t + \theta_1 \epsilon_t + \epsilon_{t+1}] = \phi_0 + \phi_1 y_t + \theta_1 \epsilon_t + \mathrm{E}_t[\epsilon_{t+1}] = \phi_0 + \phi_1 y_t + \theta_1 \epsilon_t$

(c) $\mathrm{E}_t[y_{t+2}]$

$$\begin{aligned} \mathrm{E}_t[y_{t+2}] &= \mathrm{E}_t[\phi_0 + \phi_1 y_{t+1} + \theta_1 \epsilon_{t+1} + \epsilon_{t+2}] \\ &= \phi_0 + \phi_1 \mathrm{E}_t[y_{t+1}] + \theta_1 \mathrm{E}_t[\epsilon_{t+2}] + \mathrm{E}_t[\epsilon_{t+2}] \\ &= \phi_0 + \phi_1 \mathrm{E}_t[y_{t+1}] + 0 + 0 \\ &= \phi_0 + \phi_1 (\phi_0 + \phi_1 y_t + \theta_1 \epsilon_t) \end{aligned}$$

(d) $\lim_{h\to\infty} \mathrm{E}_t[y_{t+h}]$

This is just the long-run mean, which is the answer to the first sub part, $\phi_0/1-\phi_1$. This can be seen since the h-step ahead forecast has the recursive relationship $\mathrm{E}_t[y_{t+h}] = \phi_0 + \phi_1 \mathrm{E}_t[y_{t+h-1}]$ which lead to the pattern

$$\mathrm{E}_t[y_{t+h}] = \phi_0 \sum_{i=0}^{h-2} \phi_1^i + \phi_1^{h-1} \mathrm{E}_t[y_{t+1}]$$

where the one-step forecast has been left in for simplicity. When h is large, the last term is negligible and the first term is a convergent geometric sequence.

(e) $\mathrm{V}_t[\epsilon_{t+1}] = \mathrm{E}_t\left[\epsilon_{t+1}^2\right] = \mathrm{E}_t\left[e_{t+1}^2 \sigma_{t+1}^2\right] = \sigma_{t+1}^2 \mathrm{E}_t\left[e_{t+1}^2\right] = \sigma_{t+1}^2 \times 1$

(f) $\mathrm{V}_t[y_{t+1}] = \mathrm{E}_t\left[(y_{t-1} - \mathrm{E}_t[y_{t-1}])^2\right] = \mathrm{E}_t\left[\epsilon_{t+1}^2\right] = \sigma_{t+1}^2$

(g) $\mathrm{V}_t[y_{t+2}]$

The two-step ahead forecast error is

$$\begin{aligned} y_{t+2} - \mathrm{E}_t[y_{t+2}] &= \phi_0 + \phi_1 y_{t+1} + \theta_1 \epsilon_{t+1} + \epsilon_{t+2} - \mathrm{E}_t[y_{t+2}] \\ &= \phi_0 + \phi_1 (\phi_0 + \phi_1 y_t + \theta_1 \epsilon_t + \epsilon_{t+1}) + \theta_1 \epsilon_{t+1} + \epsilon_{t+2} - \mathrm{E}_t[y_{t+2}] \\ &= \phi_0 + \phi_1 \phi_0 + \phi_1^2 y_t + \phi_1 \theta_1 \epsilon_t - \mathrm{E}_t[y_{t+2}] + \phi_1 \epsilon_{t+1} + \theta_1 \epsilon_{t+1} + \epsilon_{t+2} \\ &= (\phi_1 + \theta_1) \epsilon_{t+1} + \epsilon_{t+2} \end{aligned}$$

These two errors are White Noise and so uncorrelated. The variance is then

$$\mathrm{V}_t\left[y_{t+2}\right] = \left(\phi_1 + \theta_1\right)^2 \sigma_{t+1}^2 + \mathrm{E}_t\left[\sigma_{t+2}^2\right]$$

and

$$\mathrm{E}_t\left[\sigma_{t+2}^2\right] = \omega + \omega\alpha + \alpha^2\epsilon_t^2$$

(h) $\mathrm{V}\left[y_{t+1}\right]$

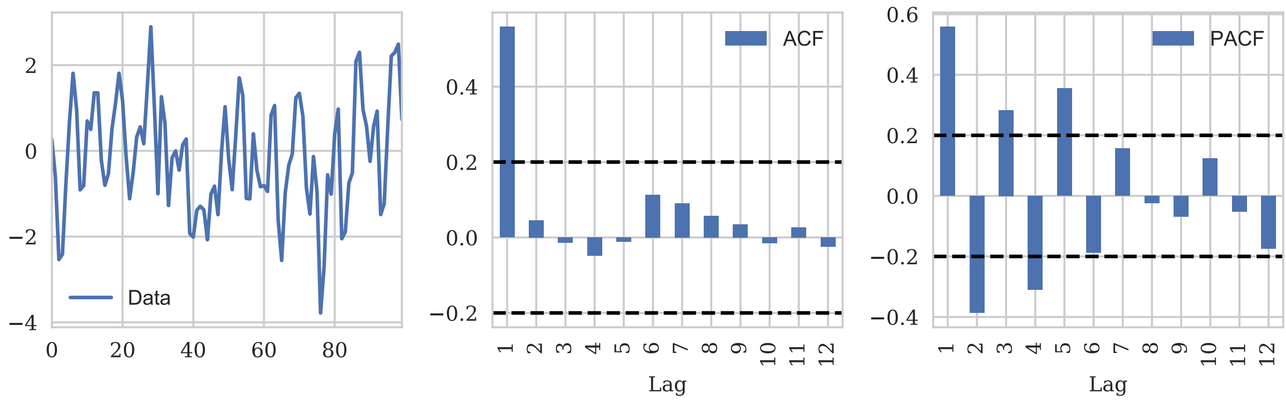
$$\begin{aligned}
\mathrm{V}\left[y_{t+1}\right] &= \mathrm{V}\left[\phi_0 + \phi_1 y_t + \theta_1\epsilon_t + \epsilon_{t+1}\right] \\
&= \phi_1^2\mathrm{V}\left[y_t\right] + \theta_1^2\mathrm{V}\left[\epsilon_t\right] + \mathrm{V}\left[\epsilon_{t+1}\right] + 2\mathrm{Cov}\left[\phi_1 y_t, \theta_1\epsilon_t\right] \\
&= \phi_1^2\mathrm{V}\left[y_t\right] + \theta_1^2\bar{\sigma}^2 + \bar{\sigma}^2 + 2\mathrm{Cov}\left[\phi_1\left(\phi_0 + \phi_1 y_{t-1} + \theta_1\epsilon_{t-1} + \epsilon_t\right), \theta_1\epsilon_t\right] \\
&= \phi_1^2\mathrm{V}\left[y_t\right] + \theta_1^2\bar{\sigma}^2 + \bar{\sigma}^2 + 2\mathrm{Cov}\left[\phi_1\epsilon_t, \theta_1\epsilon_t\right] \\
&= \phi_1^2\mathrm{V}\left[y_t\right] + \theta_1^2\bar{\sigma}^2 + \bar{\sigma}^2 + 2\phi_1\theta_1\bar{\sigma}^2 \\
\Rightarrow \mathrm{V}\left[y_{t+1}\right] &= \phi_1^2\mathrm{V}\left[y_t\right] + \left(1 + \theta_1^2 + 2\phi_1\theta_1\right)\bar{\sigma}^2 \\
\Rightarrow \mathrm{V}\left[y_{t+1}\right] &= \left(1+\theta_1^2+2\phi_1\theta_1\right)\bar{\sigma}^2/1-\phi_1^2
\end{aligned}$$

2. **Justify a reasonable model for each of these time series in Figure ?? using information in the autocorrelation and partial autocorrelation plots. In each set of plots, the leftmost panel shows that data ($T = 100$). The middle panel shows the sample autocorrelation with 95% confidence bands. The right panel shows the sample partial autocorrelation for the data with 95% confidence bands.**

   (a) **Panel (a)**
   The ACF cuts off sharply at one lag while the PACF decays slowly oscillating. This matters is consistent with an MA(1).

   (b) **Panel (b)**
   The ACF decays slowly while the PACF cuts off at one lag. This model appears to be am AR(1). It may be stationary but could also be a unit root since the sample ACF tends to decay at rate $(T - h)/T$ due to finite sample bias.

(a)



(b)

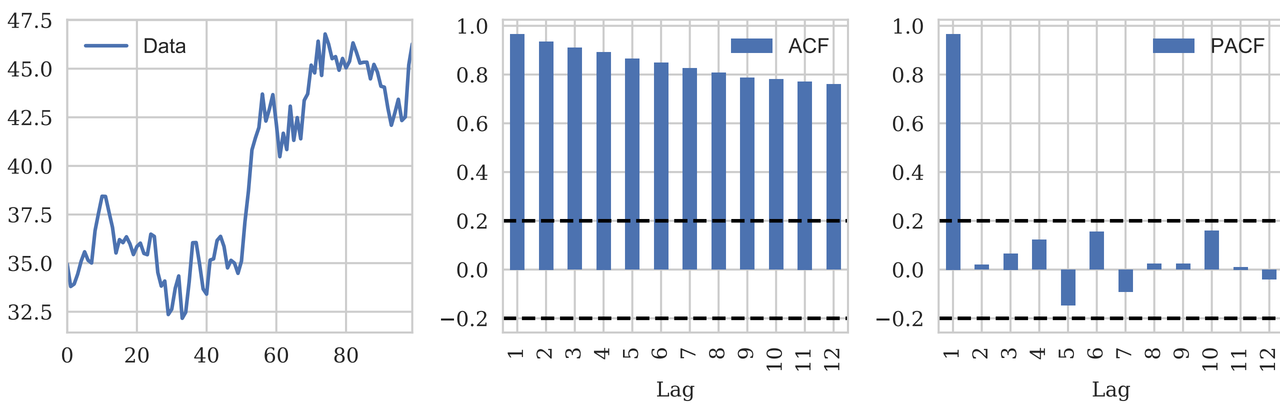


Figure 1: Plots for question