

Multivariate Volatility, Dependence and Copulas

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$$\sigma_n = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}}$$

$$D_x = \sigma^2 = M_{2x} - (M_{1x})^2$$

$$p_T(\lambda) = \frac{\lambda^c}{c!} e^{-\lambda}$$

$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\phi(v) = 4\sqrt{\frac{k^3}{\pi}} v^2 e^{-kv^2}$$

$$M_x = \sum_{i=1}^c p_i r_i$$

$$D_x = \sum_{i=1}^c p_i (x_i - M_x)^2$$

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

$$p = \lim_{N \rightarrow \infty} \frac{n}{N}$$

$$C = \frac{\epsilon \epsilon_0 S}{d}$$

$$\langle r \rangle = \frac{\langle v \rangle t}{n\sqrt{2\pi}d^2}$$



$$B = \frac{\mu_0 I}{2\pi b} (\cos \alpha_1 - \cos \alpha_2)$$

$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\Phi_2 - \Phi_1)$$

$$h\nu = A + \frac{m\nu^2}{2}$$

$$E = m_0 c^2 + \frac{m\nu^2}{2}$$

$$m = m_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$S^2 = c^2 t^2 - l^2 = i\nu$$

$$r_n = \frac{4\pi\epsilon_0 n^2 a^2}{m^2 c^2}$$



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

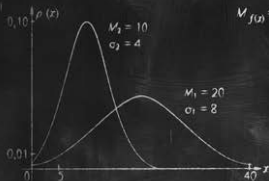
$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

$$S = v_0 t + \frac{at^2}{2}$$

$$F = G \frac{m_1 m_2}{r^2}$$

$$f(v) = 4\pi \left(\frac{m_0}{2\pi k T} \right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}$$



$$\phi(\ln x) d(\ln x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} d(\ln x) = \frac{1}{\sqrt{2\pi\sigma x}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx$$

$$\langle r \rangle = \frac{\langle v \rangle t}{n\sqrt{2\pi}d^2}$$

$$0.020 \rho(v)$$

$$E = m_0 c^2 + \frac{m\nu^2}{2}$$

$$m = m_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$S^2 = c^2 t^2 - l^2 = i\nu$$

$$r_n = \frac{4\pi\epsilon_0 n^2 a^2}{m^2 c^2}$$

$$R^2 = \frac{r^2}{(1 - \beta^2)}$$



$$D_t = \int_{-\infty}^{+\infty} (x - M_t) f(x) dx$$

- Multivariate Volatility
 - ▶ Simple Models
 - ▶ Dynamic Models
- Realized Covariance
- Dependence
 - ▶ Linear (correlation)
 - ▶ Non-linear
- Copulas

Why model Covariance?

$$\sigma_t^2 = \int_{-\infty}^{\infty} (x - M_t)^2 p(x) dx$$

- Portfolio Construction
- Portfolio Sensitivity Analysis
- Value-at-Risk
- Credit Pricing
- Correlation Trading

Preliminaries

- Returns \mathbf{r}_t are k by 1
- Use demeaned returns $\boldsymbol{\epsilon}_t = \mathbf{r}_t - \boldsymbol{\mu}_t$ where $\boldsymbol{\mu}_t$ is conditional mean, $\boldsymbol{\mu}_t = E_{t-1}[\mathbf{r}_t]$
 - ▶ In many cases of interest $\epsilon_t = r_t$
 - ▶ Horizon short and mean small relative to volatility
- Interested in conditional covariance

$$\boldsymbol{\Sigma}_t \equiv E_{t-1}[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t']$$

$$\begin{bmatrix} \epsilon_{1t}^2 & \Sigma_{1t} \epsilon_{2t} \\ \epsilon_{1t} \epsilon_{2t} & \epsilon_{2t}^2 \end{bmatrix}$$

- “Devolatilized” residuals

$$u_{i,t} = \epsilon_{i,t} / \sigma_{i,t}, i = 1, 2, \dots, k, \text{ or } \mathbf{u}_t = \boldsymbol{\epsilon}_t \oslash \boldsymbol{\sigma}_t$$

Preliminaries

- Standard covariance property

$$\text{Cov}[\mathbf{Az}] = \mathbf{A} \text{Cov}[\mathbf{z}] \mathbf{A}'$$

- Standardized residuals

$$\mathbf{e}_t = \boldsymbol{\Sigma}_t^{-\frac{1}{2}} \boldsymbol{\epsilon}_t$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_c^{1/2} \left(\boldsymbol{\Sigma}_c^{-1/2} \right)'$$
$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_c^{-1/2} \left(\boldsymbol{\Sigma}_c^{1/2} \right)'$$

- ▶ Devolatilized and decorrelated

- ▶ Since $\text{Cov}_{t-1}[\boldsymbol{\epsilon}_t] = \boldsymbol{\Sigma}_t$, $\text{Cov}_{t-1}[\boldsymbol{\Sigma}_t^{-\frac{1}{2}} \boldsymbol{\epsilon}_t] = \boldsymbol{\Sigma}_t^{-\frac{1}{2}} \boldsymbol{\Sigma}_t \boldsymbol{\Sigma}_t^{-\frac{1}{2}} = \mathbf{I}_k$

- Conditional Correlation

$$\boldsymbol{\epsilon}_t \stackrel{iid}{\sim} N(0, \boldsymbol{\Sigma}_t)$$
$$\boldsymbol{\Sigma}_t^{1/2} \boldsymbol{\epsilon}_t \sim N(0, \boldsymbol{\Sigma}_t')$$

$$\mathbf{R}_t \equiv \mathbb{E}_{t-1}[\mathbf{u}_t \mathbf{u}_t']$$

$$\boldsymbol{\Sigma}_t^{1/2} \boldsymbol{\Sigma}_t^{-1/2}$$
$$\left(\boldsymbol{\Sigma}_t^{-1/2} \boldsymbol{\Sigma}_t^{1/2} \right) \left(\boldsymbol{\Sigma}_t^{1/2} \boldsymbol{\Sigma}_t^{-1/2} \right)$$

$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$$

$$P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

$$A_n^k = \frac{n!}{(n-k)!}$$

$$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

$$\lambda_n^k = \underbrace{n \cdot n \cdot \dots \cdot n}_k = n^k$$

$$\tilde{C}_n^m = P_{m, n-m-1} = \frac{(n+m-1)!}{m!(n-1)!}$$

$$\tilde{C}_n^{r_1, r_2, \dots, r_k} = \frac{(n_1 + n_2 + \dots + n_k)!}{n_1! n_2! \dots n_k!}$$

$$C_n^c = \frac{n!}{c!(n-c)!}$$

$$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n = \sum_{k=0}^n C_n^k a^{n-k} b^k$$

$$p(B) = p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + p(B|A_3)p(A_3) + \dots + p(B|A_n)p(A_n)$$

$$p(x) = \frac{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}{p(B|A_1)p(A_1) + p(B|A_2)p(A_2) + \dots + p(B|A_n)p(A_n)}$$



$$D_x = \int_{-\infty}^{+\infty} (x - M_x)^2 \phi(x) dx$$

$$M_x = \int_{-\infty}^{+\infty} x \cdot \phi(x) dx$$

$$M_{f(x)} = \int_{-\infty}^{+\infty} f(x) \phi(x) dx$$

$$S = \sqrt{\sigma^2} = \frac{d^2}{2}$$

$$E = G \frac{m_1 m_2}{Q^2}$$

$$f(x) = A \exp\left(-\frac{m_1 x^2}{2 \sigma^2}\right) \exp\left(-\frac{m_2 x^2}{2 \sigma^2}\right)$$



Covariance Estimators

$$\sigma_{xy} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

$$D_y = \sigma_y^2 = M_y^2 - (M_y^2)^2$$

$$P_c(k) = \frac{1}{2^k} e^{-k}$$

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \phi(x) dx$$

$$M_x = \sum_{i=1}^c p_i x_i$$

$$D_x = \sum_{i=1}^c p_i (x_i - M_x)^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi c}} e^{-\frac{(x-\bar{x})^2}{2c}}$$

$$\phi(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

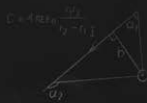
$$\phi(v) = A \sqrt{\frac{k^3}{\pi}} \sqrt{v} e^{-kv^2}$$



$$p = \frac{\log \frac{n}{N}}{N - \frac{n}{N}}$$

$$C = \frac{R \cdot S}{d}$$

$$V = \frac{(2V)}{n \sqrt{2\pi} d^2}$$



$$d = \frac{R \cdot S}{2 \sin \varphi_1}$$

$$d^2 = d_1^2 + d_2^2 + 2 d_1 d_2 \cos(\varphi_1 + \varphi_2)$$

$$nV = A + \frac{mV^2}{2}$$

$$E = m \sigma_c^2 + \frac{mV^2}{2}$$

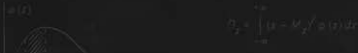
$$m = m_0 \sqrt{1 - \beta^2}$$

$$S^2 = c^2 t^2 - l^2 = in v$$

$$r_n = \frac{4 \pi \epsilon_0 n^2 n^2}{m^2 c^2}$$



Moving Average Covariance



- Simplest estimator
- Only reasonable if k is small
- Usually requires $n > k$

Definition (n -period Moving Average Covariance)

The n -period moving average covariance is defined

$$\Sigma_t = n^{-1} \sum_{i=1}^n \epsilon_{t-i} \epsilon'_{t-i}$$

Observable Factor Covariance

- Use CAP-M or APT to motivate covariance model
- Attribute all covariance to common factors

Definition (n -period Factor Covariance)

The n -period factor covariance is defined as

$$\Sigma_t = \beta' \Sigma_t^f \beta + \Omega_t$$

A handwritten diagram of a diagonal matrix Ω_t . The matrix is enclosed in large red square brackets. The diagonal elements are labeled ω_1^2 , ω_2^2 , and ω_k^2 . The off-diagonal elements are represented by circles with a slash through them, indicating they are zero.

where $\Sigma_t^f = n^{-1} \sum_{i=1}^n \mathbf{f}_{t-i} \mathbf{f}'_{t-i}$ is the n -period moving covariance of the factors,

$$\beta_t = \left(\sum_{i=1}^n \mathbf{f}_{t-i} \mathbf{f}'_{t-i} \right)^{-1} \sum_{i=1}^n \mathbf{f}_{t-i} \epsilon'_{t-i}$$

is the p by k matrix of factor loadings and Ω_t is a diagonal matrix with $\omega_{j,t}^2 = n^{-1} \sum_{i=1}^n \eta_{j,t-i}^2$ in the j^{th} diagonal position where $\eta_{i,t} = \epsilon_{i,t} - \mathbf{f}'_t \beta_i$ are the regression residuals.

Observable Factor Covariance

- Is positive semi-definite when $\lambda > p$
- Suitable for large portfolios λ
- Can be customized in portfolios with different asset classes
- In an Equity – Bond portfolio
 - ▶ 1 factor common to all assets (S&P 500)
 - ▶ 2 factors for equities (Size and Value)
 - ▶ 2 factors for bonds (Curvature and Default Premium)

Principal Component Analysis

- Split returns in the orthogonal (uncorrelated) components

$$\min_{\beta, \mathbf{F}} (kT)^{-1} \sum_{i=1}^k \sum_{t=1}^T (y_{i,t} - \mathbf{f}_t \beta_i)^2 \quad \text{subject to } \beta' \beta = \mathbf{I}_k$$

- Solution depends on eigenvalues and eigenvectors, easy to calculate
- Can order factors so that the partial R^2 are decreasing
- Factor 1 explains more than factor 2 which explains more than factor 3, etc.
- Can estimate the number of factors which are common across all assets
- *More details in notes ...*

Definition (n -period Principal Component Covariance)

The n -period principal component covariance is defined as

$$\Sigma_t = \beta_t' \Sigma_t^f \beta_t + \Omega_t$$

where $\Sigma_t^f = n^{-1} \sum_{i=1}^n \mathbf{f}_{t-i} \mathbf{f}_{t-i}'$ is the n -period moving covariance of first p principal component factors, $\hat{\beta}_t$ is the p by k matrix of principal component loadings corresponding to the first p factors, and Ω_t is a diagonal matrix with $\omega_{j,t+1}^2 = n^{-1} \sum_{i=1}^n \eta_{j,t-1}^2$ on the j^{th} diagonal where $\eta_{i,t} = r_{i,t} - \mathbf{f}_{t,i}' \beta_{i,t}$ are the residuals from a p -factor principal component analysis.

- Use principal components in place of observable factors
- Same advantage as observable factor covariance
 - ▶ Additional advantage that do not need factors
- Disadvantage that not as easy to implement in a structured setting
- Identical to moving average covariance if all factors used

$$\hat{\rho}_t = \frac{n!}{(n-2)!}$$



$$\sigma_t^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

- Assume that all correlations are identical

$$\sigma_{ij,t} = \rho \sigma_{i,t} \sigma_{j,t}$$

Definition (n -period Moving Average Equicorrelation Covariance)

The n -period moving average equicorrelation covariance is defined as

$$\Sigma_t = \begin{bmatrix} \sigma_{1,t}^2 & \rho_t \sigma_{1,t} \sigma_{2,t} & \rho_t \sigma_{1,t} \sigma_{3,t} & \cdots & \rho_t \sigma_{1,t} \sigma_{k,t} \\ \rho_t \sigma_{1,t} \sigma_{2,t} & \sigma_{2,t}^2 & \rho_t \sigma_{2,t} \sigma_{3,t} & \cdots & \rho_t \sigma_{2,t} \sigma_{k,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_t \sigma_{1,t} \sigma_{k,t} & \rho_t \sigma_{2,t} \sigma_{k,t} & \rho_t \sigma_{3,t} \sigma_{k,t} & \cdots & \sigma_{k,t}^2 \end{bmatrix}$$

where $\sigma_{j,t}^2 = n^{-1} \sum_{i=1}^n \epsilon_{j,t}^2$ and ρ_t is estimated using one of the estimators below.

Equicorrelation

- Moment or maximum likelihood estimator for ρ
- Moment:

$$\begin{aligned} E[\epsilon_{p,t}^2] &= k^{-2} \sum_{j=1}^k \sigma_{j,t}^2 + 2k^{-2} \sum_{o=1}^k \sum_{q=o+1}^k \rho \sigma_{o,t} \sigma_{q,t} \\ &= k^{-2} \sum_{j=1}^k \sigma_{j,t}^2 + 2\rho k^{-2} \sum_{o=1}^k \sum_{q=o+1}^k \sigma_{o,t} \sigma_{q,t} \end{aligned}$$

- Estimator exploits this structure

$$\rho_t = \frac{\sigma_{p,t}^2 - k^{-2} \sum_{j=1}^k \sigma_{j,t}^2}{2k^{-2} \sum_{o=1}^k \sum_{q=o+1}^k \sigma_{o,t} \sigma_{q,t}}.$$

- $\sigma_{j,t}^2$ are the volatilities of the individual assets
- Only appropriate for homogeneous portfolios

Factor Models on the S&P 500

- Daily data on S&P 500 constituents from January 1, 1999 – December 31, 2008
- Return only included if present in relevant sample
 - ▶ Full sample
 - ▶ Rolling 252-day sample, centered at sample mid-point
- Full Sample PCA

$k = 194$	1	2	3	4	5	6
Partial R^2	0.263	0.039	0.031	0.023	0.019	0.016
Cumulative R^2	0.263	0.302	0.333	0.356	0.375	0.391

- Full Sample Correlation

Equicorrelation	1-Factor R^2 (S&P 500)	3-Factor R^2 (Fama-French)
0.255	0.236	0.267

Rolling Window Correlations

$$\sigma_p^2 = \int_{-\infty}^{\infty} (x - \mu_p)^2 f(x) dx$$

