Multivariate Volatility, Dependence and Copulas

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Overview

- Multivariate Volatility
  - Simple Models
  - Dynamic Models
- Realized Covariance
- Dependence
  - Linear (correlation)
  - Non-linear
- Copulas

Why model Covariance?

- Portfolio Construction
- Portfolio Sensitivity Analysis
- Value-at-Risk
- Credit Pricing
- Correlation Trading

Preliminaries

- Returns $\mathbf{r}_t$ are k by 1
- Use demeaned returns $\boldsymbol{\epsilon}_t = \mathbf{r}_t - \boldsymbol{\mu}_t$ where $\boldsymbol{\mu}_t$ is conditional mean, $\boldsymbol{\mu}_t = \mathrm{E}_{t-1}\left[\mathbf{r}_t\right]$
  - In many cases of interest $\boldsymbol{\epsilon}_t = \mathbf{r}_t$
  - Horizon short and mean small relative to volatility
- Interested in conditional covariance

$$\boldsymbol{\Sigma}_t \equiv \mathrm{E}_{t-1}\left[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t'\right]$$

- "Devolatilized" residuals

$$u_{i,t} = \epsilon_{i,t}/\sigma_{i,t}, i = 1, 2, \ldots, k, \text{ or } \mathbf{u}_t = \boldsymbol{\epsilon}_t \oslash \boldsymbol{\sigma}_t$$

Preliminaries

- Standard covariance property

$$\text{Cov}\left[\mathbf{Az}\right] = \mathbf{A}\text{Cov}\left[\mathbf{z}\right]\mathbf{A}'$$

- Standardized residuals

$$\mathbf{e}_t = \boldsymbol{\Sigma}_t^{-\frac{1}{2}}\boldsymbol{\epsilon}_t$$

  - Devolatilized and decorrelated
  - Since $\text{Cov}_{t-1}\left[\boldsymbol{\epsilon}_t\right] = \boldsymbol{\Sigma}_t$, $\text{Cov}_{t-1}\left[\boldsymbol{\Sigma}_t^{-\frac{1}{2}}\boldsymbol{\epsilon}_t\right] = \boldsymbol{\Sigma}_t^{-\frac{1}{2}}\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t^{-\frac{1}{2}} = \mathbf{I}_k$

- Conditional Correlation

$$\mathbf{R}_t \equiv \text{E}_{t-1}\left[\mathbf{u}_t\mathbf{u}_t'\right]$$

Covariance Estimators

Moving Average Covariance

- Simplest estimator
- Only reasonable is k is small
- Usually requires $n > k$

Definition (n-period Moving Average Covariance)

The n-period moving average covariance is defined

$$\boldsymbol{\Sigma}_t = n^{-1} \sum_{i=1}^{n} \boldsymbol{\epsilon}_{t-i} \boldsymbol{\epsilon}'_{t-i}$$

Observable Factor Covariance

- Use CAP-M or APT to motivate covariance model
- Attribute all covariance to common factors

Definition (n-period Factor Covariance)

The n-period factor covariance is defined as

$$\boldsymbol{\Sigma}_t = \boldsymbol{\beta}' \boldsymbol{\Sigma}_t^f \boldsymbol{\beta} + \boldsymbol{\Omega}_t$$

where $\boldsymbol{\Sigma}_t^f = n^{-1} \sum_{i=1}^{n} \mathbf{f}_{t-i} \mathbf{f}'_{t-i}$ is the n-period moving covariance of the factors,

$$\boldsymbol{\beta}_t = \left(\sum_{i=1}^{n} \mathbf{f}_{t-i} \mathbf{f}'_{t-i} \right)^{-1} \sum_{i=1}^{n} \mathbf{f}_{t-i} \boldsymbol{\epsilon}'_{t-i}$$

is the p by k matrix of factor loadings and $\boldsymbol{\Omega}_t$ is a diagonal matrix with $\omega_{j,t}^2 = n^{-1} \sum_{i=1}^{n} \eta_{j,t-i}^2$ in the j[th] diagonal position where $\eta_{i,t} = \epsilon_{i,t} - \mathbf{f}'_t \boldsymbol{\beta}_i$ are the regression residuals.

Observable Factor Covariance

- Is positive semi-definite when $k > p$
- Suitable for large portfolios
- Can be customized in portfolios with different asset classes
- In an Equity – Bond portfolio
  - 1 factor common to all assets (S&P 500)
  - 2 factors for equities (Size and Value)
  - 2 factors for bonds (Curvature and Default Premium)

Principal Component Analysis

- Split returns in the orthogonal (uncorrelated) components

$$\min_{\boldsymbol{\beta},\mathbf{F}} (kT)^{-1} \sum_{i=1}^{k} \sum_{t=1}^{T} (y_{i,t} - \mathbf{f}_t \boldsymbol{\beta}_i)^2 \text{ subject to } \boldsymbol{\beta}' \boldsymbol{\beta} = \mathbf{I}_k$$

- Solution depends on eigenvalues and eigenvectors, easy to calculate
- Can order factors so that the partial R^2 are decreasing
- Factor 1 explains more than factor 2 which explains more than factor 3, etc.
- Can estimate the number of factors which are common across all assets
- *More details in notes ...*

Principal Component Covariance

Definition (n-period Principal Component Covariance)

The n-period principal component covariance is defined as

$$\boldsymbol{\Sigma}_t = \boldsymbol{\beta}_t' \boldsymbol{\Sigma}_t^f \boldsymbol{\beta}_t + \boldsymbol{\Omega}_t$$

where $\boldsymbol{\Sigma}_t^f = n^{-1} \sum_{i=1}^{n} \mathbf{f}_{t-i} \mathbf{f}_{t-i}'$ is the n-period moving covariance of first p principal component factors, $\hat{\boldsymbol{\beta}}_t$ is the p by k matrix of principal component loadings corresponding to the first p factors, and $\boldsymbol{\Omega}_t$ is a diagonal matrix with $\omega_{j,t+1}^2 = n^{-1} \sum_{i=1}^{n} \eta_{j,t-1}^2$ on the j$^{\text{th}}$ diagonal where $\eta_{i,t} = r_{i,t} - \mathbf{f}_t' \boldsymbol{\beta}_{i,t}$ are the residuals from a p-factor principal component analysis.

- Use principal components in place of observable factors
- Same advantage as observable factor covariance
  - Additional advantage that do not need factors
- Disadvantage that not as easy to implement in a structured setting
- Identical to moving average covariance if all factors used

Equicorrelation

- Assume that all correlations are identical

$$\sigma_{ij,t} = \rho\sigma_{i,t}\sigma_{j,t}$$

Definition (n-period Moving Average Equicorrelation Covariance)

The n-period moving average equicorrelation covariance is defined as

$$\boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma^2_{1,t} & \rho_t\sigma_{1,t}\sigma_{2,t} & \rho_t\sigma_{1,t}\sigma_{3,t} & \ldots & \rho_t\sigma_{1,t}\sigma_{k,t} \\ \rho_t\sigma_{1,t}\sigma_{2,t} & \sigma^2_{2,t} & \rho_t\sigma_{2,t}\sigma_{3,t} & \ldots & \rho_t\sigma_{2,t}\sigma_{k,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_t\sigma_{1,t}\sigma_{k,t} & \rho_t\sigma_{2,t}\sigma_{k,t} & \rho_t\sigma_{3,t}\sigma_{k,t} & \ldots & \sigma^2_{k,t} \end{bmatrix}$$

where $\sigma^2_{j,t} = n^{-1}\sum_{i=1}^{n}\epsilon^2_{j,t}$ and ρ_t is estimated using one of the estimators below.

Equicorrelation

- Moment or maximum likelihood estimator for ρ
- Moment:

$$\begin{aligned}\mathrm{E}[\epsilon_{p,t}^2] &= k^{-2}\sum_{j=1}^{k}\sigma_{j,t}^2 + 2k^{-2}\sum_{o=1}^{k}\sum_{q=o+1}^{k}\rho\sigma_{o,t}\sigma_{q,t} \\ &= k^{-2}\sum_{j=1}^{k}\sigma_{j,t}^2 + 2\rho k^{-2}\sum_{o=1}^{k}\sum_{q=o+1}^{k}\sigma_{o,t}\sigma_{q,t}\end{aligned}$$

- Estimator exploits this structure

$$\rho_t = \frac{\sigma_{p,t}^2 - k^{-2}\sum_{j=1}^{k}\sigma_{j,t}^2}{2k^{-2}\sum_{o=1}^{k}\sum_{q=o+1}^{k}\sigma_{o,t}\sigma_{q,t}}.$$

- $\sigma_{j,t}^2$ are the volatilities of the individual assets
- Only appropriate for homogeneous portfolios

Factor Models on the S&P 500

- Daily data on S&P 500 constituents from January 1, 1999 – December 31, 2008
- Return only included if present in relevant sample
  - Full sample
  - Rolling 252-day sample, centered at sample mid-point
- Full Sample PCA

k = 194	1	2	3	4	5	6
Partial R^2	0.263	0.039	0.031	0.023	0.019	0.016
Cumulative R^2	0.263	0.302	0.333	0.356	0.375	0.391

- Full Sample Correlation

Equicorrelation	1-Factor R^2 (S&P 500)	3-Factor R^2 (Fama-French)
0.255	0.236	0.267

Rolling Window Correlations