Financial Econometrics
HT Week 6 Assignment Answers

February 2021

Exercise 7.25

If $\ln RV_t$ is modeled as a HAR

$$\ln RV_t = 0.1 + 0.4\ln RV_{t-1} + 0.3\ln RV_{t-1:5} + 0.22\ln RV_{t-1:22} + \varepsilon_t$$

where $\varepsilon_t \overset{\text{i.i.d.}}{\sim} N(0,\sigma^2)$ where $\ln RV_{t-1:h} = h^{-1}\sum_{i=1}^{h}\ln RV_{t-i}$ is the average of h lags of $\ln RV$.

1. **What is $\mathrm{E}_t[\ln RV_{t+1}]$?**

$$\begin{aligned}\mathrm{E}_t[\ln RV_{t+1}] &= 0.1 + 0.4\mathrm{E}_t[\ln RV_t] + 0.3\mathrm{E}_t\left[\ln RV_{(t+1)-1:5}\right] + 0.22\mathrm{E}_t\left[\ln RV_{(t+1)-1:22}\right] + \mathrm{E}_t[\varepsilon_{t+1}]\\ &= 0.1 + 0.4\ln RV_t + 0.3\ln RV_{(t+1)-1:5} + 0.22\ln RV_{(t+1)-1:22}\end{aligned}$$

2. **What is $\mathrm{E}_t[\ln RV_{t+2}]$?**

$$\begin{aligned}\ln RV_{t+2} &= 0.1 + 0.4\ln RV_{t+1} + 0.3\ln RV_{(t+2)-1:5} + 0.22\ln RV_{(t+2)-1:22} + \varepsilon_{t+2}\\ &= 0.1 + 0.4\ln RV_{t+1} + {}^{0.3}/_{5}(\ln RV_{t+1} + \ln RV_t + \ldots + \ln RV_{t-3}) + {}^{0.22}/_{22}(\ln RV_{t+1} + \ln RV_t + \ldots + \ln RV_{t-20}) + \varepsilon\\ &= 0.1 + 0.47\ln RV_{t+1} + 0.07(\ln RV_t + \ldots + \ln RV_{t-3}) + 0.01(\ln RV_{t-4} + \ldots + \ln RV_{t-20}) + \varepsilon_{t+2}\\ \mathrm{E}_t[\ln RV_{t+2}] &= 0.1 + 0.47\mathrm{E}_t[\ln RV_{t+1}] + 0.07(\mathrm{E}_t[\ln RV_t] + \ldots + \mathrm{E}_t[\ln RV_{t-3}]) + 0.01(\mathrm{E}_t[\ln RV_{t-4}] + \ldots + \mathrm{E}_t[\ln RV_{t-20}]) + \mathrm{E}\\ &= 0.1 + 0.47\mathrm{E}_t[\ln RV_{t+1}] + 0.07(\ln RV_t + \ldots + \ln RV_{t-3}) + 0.01(\ln RV_{t-4} + \ldots + \ln RV_{t-20})\end{aligned}$$

   where $\mathrm{E}_t[\ln RV_{t+1}]$ was computed in the previous step.

3. **What is $\lim_{h\to\infty}\mathrm{E}_t[\ln RV_{t+h}]$?**
   Here we use the fact that the long run limit is just he unconditional mean. We should check that this model is covariance stationary (at least in an online exam), and it is since the largest root of the characteristic polynomial is 0.972. With this in mind, the long-run-mean is just

$$\frac{0.1}{1-\sum_{i=1}^{22}\phi_i} = \frac{0.1}{1-0.92} = 1.25$$

4. **What is the conditional distribution of the 2-step forecast error, $\ln RV_{t+2} - \mathrm{E}_t[\ln RV_{t+2}]$?**
   We can start by writing the model as a standard AR(22)

$$\ln RV_{t+2} = 0.1 + 0.47\ln RV_{t+1} + 0.06\mathrm{lnRV}_t + \ldots + 0.06\ln RV_{t-3} + 0.01\ln RV_{t-4} + \ldots + 0.01\ln RV_{t-20} + \varepsilon_{t+2}$$

and then substitute in for $\ln RV_{t+1}$

$$
\begin{aligned}
\ln RV_{t+2} &= 0.1 + 0.47\left(0.1 + 0.47 \ln RV_t + 0.06 \ln \mathrm{RV}_{t-1} + \ldots + 0.06 \ln RV_{t-4} + 0.01 \ln RV_{t-5} + \ldots + 0.01 \ln RV_{t-21} + \varepsilon_{t+1}\right) + \\
&= 0.147 + 0.2809 \ln RV_t + \sum_{i=1}^{3} 0.0882 \ln RV_{t-i} + 0.0382 \ln RV_{t-4} + \sum_{j=5}^{21} 0.0147 \ln RV_{t+j} + 0.0047 \ln RV_{t-22} + 0.47\varepsilon_t
\end{aligned}
$$

so that the 2-step forecast error is $0.47\varepsilon_{t-1} + \varepsilon_{t+2}$. The variance of the error is then $\left(1 + 0.47^2\right)\sigma^2$ since the error is i.i.d. normal, and the distribution is

$$N\left(0, \left(1 + 0.47^2\right)\sigma^2\right)$$

5. **What is** $\mathrm{E}_t\left[RV_{t+1}\right]$**?**
The log is normal, and so the exponential of the log is Log-Normal. We can use properties of the Log-Normal to compute the expectation, which is

$$\mathrm{E}_t\left[RV_{t+1}\right] = \exp\left(\mathrm{E}_t\left[\ln RV_{t+1}\right] + \sigma^2/2\right)$$

6. **What is** $\mathrm{E}_t\left[RV_{t+2}\right]$**?**
Here we use the error distribution previously computed, and so

$$\mathrm{E}_t\left[RV_{t+1}\right] = \exp\left(\mathrm{E}_t\left[\ln RV_{t+2}\right] + \left(1+0.47^2\right)\sigma^2/2\right)$$