Financial Econometrics
HT Week 5 Assignment Answers

February 2021

Exercise 7.1

What is Realized Variance and why is it useful?

Realized variance is an estimator based on high-frequency data. Typically it is constructed using ultra-high-frequency data, either transactions or quotes. It can also be constructed by treating daily data as "high frequency" in the context of modeling monthly returns. Under some assumptions, realized variance consistently estimates the integrated variance as the number of samples in the day increases.

Realized variance is estimated from the sum of m high-frequency returns

$$RV_t^{(m)} = \sum_{i=1}^{m} r_{i,t}^2$$

without demeaning or dividing by m. Is useful as a measure of the variance on day t since it is a nonparametric estimator that is consistent under some assumptions, which may not be satisfied. Even when m can only be set to a moderate value, e.g., 78 for 5-miunte returns, it is still a much better measure of the variance than the 1-sample $RV_t^{(1)}$ which is just the squared return. It is used in both forecasting variance and for evaluating the specification of variance forecasts in GMZ regressions.

Exercise 7.24

Suppose $\{Y_t\}$ is covariance stationary and can be described by the following process:

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t \\ \varepsilon_t &= \sigma_t e_t \\ \sigma_t^2 &= \omega + \alpha \varepsilon_{t-1}^2 \\ e_t &\stackrel{\text{i.i.d.}}{\sim} N(0,1) \end{aligned}$$

what are the values of the following quantities:

i. $\mathrm{E}_t[Y_{t+1}]$

$$\begin{aligned} \mathrm{E}_t[Y_{t+1}] &= \mathrm{E}_t[\phi_1 Y_t + \theta_1 \varepsilon_t + \varepsilon_{t+1}] \\ &= \phi_1 Y_t + \theta_1 \varepsilon_t + \mathrm{E}_t[\varepsilon_{t+1}] \\ &= \phi_1 Y_t + \theta_1 \varepsilon_t \end{aligned}$$

ii. $\mathrm{E}_t[Y_{t+2}]$

$$\begin{aligned} \mathrm{E}_t[Y_{t+2}] &= \mathrm{E}_t[\phi_1 Y_{t+1} + \theta_1 \varepsilon_{t+1} + \varepsilon_{t+2}] \\ &= \phi_1 \mathrm{E}_t[Y_{t+1}] + \theta_1 \mathrm{E}_t[\varepsilon_{t+1}] + \mathrm{E}_t[\varepsilon_{t+2}] \\ &= \phi_1 \mathrm{E}_t[Y_{t+1}] \\ &= \phi_1 (\phi_1 Y_t + \theta_1 \varepsilon_t) \\ &= \phi_1^2 Y_t + \phi_1 \theta_1 \varepsilon_t \end{aligned}$$

iii. $\lim_{h\to\infty} \mathrm{E}_t[Y_{t+h}]$
The forecasts will always follow the pattern that $\mathrm{E}_t[Y_{t+h}] = \phi_1 \mathrm{E}_t[Y_{t+h-1}]$ and so the h-step ahead forecast is $\phi_1^{h-1}(\phi_1 Y_t + \theta_1 \varepsilon_t)$. Since $|\phi_1| < 1$, the limit of this forecast is 0.

iv. $\mathrm{V}_t[\varepsilon_{t+1}]$
$\mathrm{V}_t[\varepsilon_{t+1}] = \mathrm{E}_t\left[\varepsilon_{t+1}^2\right]$ since it is mean 0. $\mathrm{E}_t\left[\varepsilon_{t+1}^2\right] = \sigma_{t+1}^2$ since $\mathrm{E}_t\left[\varepsilon_{t+1}^2\right] = \mathrm{E}_t\left[e_{t+1}^2 \sigma_{t+1}^2\right] = \sigma_{t+1}^2 \mathrm{E}_t\left[e_{t+1}^2\right]$ and $\mathrm{E}_t\left[e_{t+1}^2\right] = 1$. Finally $\sigma_{t+1}^2 = \omega + \alpha \varepsilon_t^2$ since these values are all known at time t.

v. $\mathrm{V}_t[Y_{t+1}]$
The one-step ahead forecast error is $Y_{t+1} - \mathrm{E}_t[Y_{t+1}] = \varepsilon_{t+1}$, and so $\mathrm{V}_t[Y_{t+1}] = \mathrm{V}_t[\varepsilon_{t+1}]$

vi. $\mathrm{V}_t[Y_{t+2}]$
The two-step ahead forecast error is

$$\begin{aligned} Y_{t+2} - \mathrm{E}_t[Y_{t+2}] &= Y_{t+2} - \phi_1^2 Y_t - \phi_1 \theta \varepsilon_t \\ &= \phi_1 Y_{t+1} + \theta_1 \varepsilon_{t+1} + \varepsilon_{t+2} - \phi_1^2 Y_t - \phi_1 \theta \varepsilon_t \\ &= \phi_1 (\phi_1 Y_t + \theta_1 \varepsilon_t + \varepsilon_{t+1}) + \theta_1 \varepsilon_{t+1} + \varepsilon_{t+2} - \phi_1^2 Y_t - \phi_1 \theta \varepsilon_t \\ &= \phi_1 \varepsilon_{t+1} + \theta_1 \varepsilon_{t+1} + \varepsilon_{t+2} \\ &= (\phi_1 + \theta_1)\varepsilon_{t+1} + \varepsilon_{t+2} \end{aligned}$$

The two shocks are serially uncorrelated, and so the variance of the sum is the sum of the variances, and

$$V_t\left[(\phi_1+\theta_1)\,\varepsilon_{t+1}+\varepsilon_{t+2}\right]=(\phi_1+\theta_1)^2\,V_t\left[\varepsilon_{t+1}\right]+V_t\left[\varepsilon_{t+2}\right].$$

Finally, the two-step ahead variance $V_t\left[\varepsilon_{t+2}\right]=E_t\left[\sigma^2_{t+2}\right]$ is

$$\begin{aligned}E_t\left[\sigma^2_{t+2}\right]&=E_t\left[\omega+\alpha_1\varepsilon^2_{t+1}\right]\\&=\omega+\alpha_1 E_t\left[\varepsilon^2_{t+1}\right]\\&=\omega+\alpha_1\left(\omega+\alpha_1\varepsilon^2_t\right).\end{aligned}$$

vii. $\lim_{h\to\infty} V_t\left[\varepsilon_{t+h}\right]$
The h-step ahead variance in an ARCH(1) model is $\omega\sum_{i=0}^{h-1}\alpha_1+\alpha_1^h\varepsilon_t^2$. When h is large, α^h is negligible and the forecast is $\omega/1-\alpha_1$ since the first term is a the sum of a convergent geometric sequence.