Financial Econometrics
HT Week 2 Assignment Answers

February 2021

Exercise 4.25

What are the 1-step and 2-step forcasts $\mathrm{E}_t\left[Y_{t+h}\right]$ from the models:

1. $Y_t = \phi_0 + \delta t + \varepsilon_t$

$$\begin{aligned} \mathrm{E}_t\left[Y_{t+1}\right] &= \mathrm{E}_t\left[\phi_0 + \delta\left(t+1\right) + \varepsilon_{t+1}\right] \\ &= \phi_0 + \delta\left(t+1\right) + \mathrm{E}_t\left[\varepsilon_{t+1}\right] \\ &= \phi_0 + \delta\left(t+1\right) \end{aligned}$$

$$\begin{aligned} \mathrm{E}_t\left[Y_{t+2}\right] &= \mathrm{E}_t\left[\phi_0 + \delta\left(t+2\right) + \varepsilon_{t+2}\right] \\ &= \phi_0 + \delta\left(t+2\right) + \mathrm{E}_t\left[\varepsilon_{t+2}\right] \\ &= \phi_0 + \delta\left(t+2\right) \end{aligned}$$

2. $Y_t = \phi_0 + \delta t + \phi_1 Y_{t-1} + \varepsilon_t$

$$\begin{aligned} \mathrm{E}_t\left[Y_{t+1}\right] &= \mathrm{E}_t\left[\phi_0 + \delta\left(t+1\right) + \phi_1 Y_t + \varepsilon_{t+1}\right] \\ &= \phi_0 + \delta\left(t+1\right) + \phi_1 \mathrm{E}_t\left[Y_t\right] + \mathrm{E}_t\left[\varepsilon_{t+1}\right] \\ &= \phi_0 + \delta\left(t+1\right) + \phi_1 Y_t \end{aligned}$$

$$\begin{aligned} \mathrm{E}_t\left[Y_{t+2}\right] &= \mathrm{E}_t\left[\phi_0 + \delta\left(t+2\right) + \phi_1 Y_{t+1} + \varepsilon_{t+2}\right] \\ &= \phi_0 + \delta\left(t+2\right) + \phi_1 \mathrm{E}_t\left[Y_{t+1}\right] + \mathrm{E}_t\left[\varepsilon_{t+2}\right] \\ &= \phi_0 + \delta\left(t+2\right) + \phi_1\left(\phi_0 + \delta\left(t+1\right) + \phi_1 Y_t\right) \\ &= \phi_0 + \phi_1\phi_0 + \delta\left(t+2\right) + \phi_1\delta\left(t+1\right) + \phi_1^2 Y_t \end{aligned}$$

3. $Y_t = \phi_0 + \delta_1 t + \delta_2 t^2 + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

$$\begin{aligned}
\mathrm{E}_t[Y_{t+1}] &= \mathrm{E}_t\left[\phi_0 + \delta_1(t+1) + \delta_2(t+1)^2 + \theta_1\varepsilon_t + \varepsilon_{t+1}\right] \\
&= \phi_0 + \delta_1(t+1) + \delta_2(t+1)^2 + \theta_1\mathrm{E}_t[\varepsilon_t] + \mathrm{E}_t[\varepsilon_{t+1}] \\
&= \phi_0 + \delta_1(t+1) + \delta_2(t+1)^2 + \theta_1\varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}_t[Y_{t+2}] &= \mathrm{E}_t\left[\phi_0 + \delta_1(t+2) + \delta_2(t+2)^2 + \theta_1\varepsilon_{t+1} + \varepsilon_{t+2}\right] \\
&= \phi_0 + \delta_1(t+2) + \delta_2(t+2)^2 + \theta_1\mathrm{E}_t[\varepsilon_{t+1}] + \mathrm{E}_t[\varepsilon_{t+2}] \\
&= \phi_0 + \delta_1(t+2) + \delta_2(t+2)^2
\end{aligned}$$

4. $\ln Y_t = \phi_0 + \delta t + \varepsilon_t$**,** $\varepsilon \overset{\text{i.i.d.}}{\sim} N(0,\sigma^2)$ **(use properties of Lognormal random variables).**
Recall if $W \sim LogNormal(\mu,\sigma^2)$ then $\mathrm{E}[W] = \exp(\mu + \sigma^2/2)$.

$$\begin{aligned}
\mathrm{E}_t[\ln Y_{t+1}] &= \mathrm{E}_t[\phi_0 + \delta(t+1) + \varepsilon_{t+1}] \\
&= \phi_0 + \delta(t+1) + \mathrm{E}_t[\varepsilon_{t+1}] \\
&= \phi_0 + \delta(t+1)
\end{aligned}$$

$$\begin{aligned}
\mathrm{V}_t[\ln Y_{t+1}] &= \mathrm{E}_t\left[(\phi_0 + \delta(t+1) + \varepsilon_{t+1} - \mathrm{E}_t[\ln Y_{t+1}])^2\right] \\
&= \mathrm{E}_t\left[\varepsilon_{t+1}^2\right] \\
&= \sigma^2
\end{aligned}$$

$$\mathrm{E}_t[Y_{t+1}] = \exp(\phi_0 + \delta(t+1) + \sigma^2/2)$$

$$\begin{aligned}
\mathrm{E}_t[\ln Y_{t+2}] &= \mathrm{E}_t[\phi_0 + \delta(t+2) + \varepsilon_{t+2}] \\
&= \phi_0 + \delta(t+2) + \mathrm{E}_t[\varepsilon_{t+2}] \\
&= \phi_0 + \delta(t+2)
\end{aligned}$$

$$\begin{aligned}
\mathrm{V}_t[\ln Y_{t+2}] &= \mathrm{E}_t\left[(\phi_0 + \delta(t+2) + \varepsilon_{t+2} - \mathrm{E}_t[\ln Y_{t+2}])^2\right] \\
&= \mathrm{E}_t\left[\varepsilon_{t+2}^2\right] \\
&= \sigma^2
\end{aligned}$$

$$\mathrm{E}_t[Y_{t+2}] = \exp(\phi_0 + \delta(t+2) + \sigma^2/2)$$

5. $\ln Y_t = \ln Y_{t-1} + \varepsilon_t$, $\varepsilon \overset{\text{i.i.d.}}{\sim} N\left(0, \sigma^2\right)$ **(use properties of Lognormal random variables)**

$$\begin{aligned}
\mathrm{E}_t\left[\ln Y_{t+1}\right] &= \mathrm{E}_t\left[\ln Y_t + \varepsilon_{t+1}\right] \\
&= \ln Y_t + \mathrm{E}_t\left[\varepsilon_{t+1}\right] \\
&= \ln Y_t
\end{aligned}$$

$$\begin{aligned}
\mathrm{V}_t\left[\ln Y_{t+1}\right] &= \mathrm{E}_t\left[\left(\ln Y_t + \varepsilon_{t+1} - \mathrm{E}_t\left[\ln Y_{t+1}\right]\right)^2\right] \\
&= \mathrm{E}_t\left[\varepsilon_{t+1}^2\right] \\
&= \sigma^2
\end{aligned}$$

$$\mathrm{E}_t\left[Y_{t+1}\right] = \exp\left(\ln Y_t + \sigma^2/2\right)$$

$$\begin{aligned}
\mathrm{E}_t\left[\ln Y_{t+2}\right] &= \mathrm{E}_t\left[\ln Y_{t+1} + \varepsilon_{t+2}\right] \\
&= \mathrm{E}\left[\ln Y_{t+1}\right] + \mathrm{E}_t\left[\varepsilon_{t+2}\right] \\
&= \ln Y_t
\end{aligned}$$

$$\begin{aligned}
\mathrm{V}_t\left[\ln Y_{t+2}\right] &= \mathrm{E}_t\left[\left(\ln Y_{t+1} + \varepsilon_{t+2} - \mathrm{E}_t\left[\ln Y_{t+2}\right]\right)^2\right] \\
&= \mathrm{E}_t\left[\left(\ln Y_t + \varepsilon_{t+2} + \varepsilon_{t+2} - \mathrm{E}_t\left[\ln Y_{t+2}\right]\right)^2\right] \\
&= \mathrm{E}_t\left[\left(\varepsilon_{t+2} + \varepsilon_{t+2}\right)^2\right] \\
&= 2\sigma^2 \text{ since } \varepsilon_t \overset{\text{i.i.d.}}{\sim} N\left(0, \sigma^2\right)
\end{aligned}$$

$$\mathrm{E}_t\left[Y_{t+2}\right] = \exp\left(\ln Y_t + \sigma^2\right)$$

Exercise 4.26

Write the following models using both lag notation and as the standard ARMA representation where Y_t is the left-hand-side variable:

1. SARIMA$(1,0,0)\times(1,0,0,4)$

$$
\begin{aligned}
(1-\phi_1 L)\left(1-\phi_s L^4\right) Y_t &= \phi_0 + \varepsilon_t \\
Y_t &= \phi_0 + \phi_1 Y_{t-1} + \phi_s Y_{t-4} - \phi_1 \phi_s Y_{t-5} + \varepsilon_t
\end{aligned}
$$

2. SARIMA$(0,0,2)\times(1,1,0,12)$

$$
\begin{aligned}
\left(1-L^{12}\right)\left(1-\phi_s L^{12}\right) Y_t &= \phi_0 + \left(1+\theta_1 L+\theta_2 L^2\right)\varepsilon_t \\
\Delta_{12} Y_t &= \phi_0 + \phi_s \Delta_{12} Y_{t-12} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \varepsilon_t \\
Y_t &= \phi_0 + \phi_s Y_{t-1} + Y_{t-12} - \phi_s Y_{t-13} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \varepsilon_t
\end{aligned}
$$

3. SARIMA$(2,0,2)\times(0,0,0,0)$

$$
\begin{aligned}
\left(1-\phi_1 L-\phi_2 L^2\right) Y_t &= \phi_0 + \left(1+\theta_1 L+\theta_2 L^2\right)\varepsilon_t \\
Y_t &= \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \varepsilon_t
\end{aligned}
$$

4. SARIMA$(1,2,1)\times(0,0,0,0)$

$$
\begin{aligned}
(1-\phi_1 L)(1-L)^2 Y_t &= \phi_0 + (1+\theta_1 L)\varepsilon_t \\
\Delta^2 Y_t &= \phi_0 + \phi_1 \Delta^2 Y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t \\
Y_t - 2Y_{t-1} + Y_{t-2} &= \phi_0 + \phi_1\left(Y_{t-1} - 2Y_{t-2} + Y_{t-3}\right) + \theta_1 \varepsilon_{t-1} + \varepsilon_t \\
Y_t &= \phi_0 + (\phi_1+2) Y_{t-1} + (-2\phi_1-1) Y_{t-2} + \phi_1 Y_{t-3} + \theta_1 \varepsilon_{t-1} + \varepsilon_t
\end{aligned}
$$

5. SARIMA$(0,0,0)\times(1,1,1,24)$

$$
\begin{aligned}
\left(1-\phi_s L^{24}\right)\left(1-L^{24}\right) Y_t &= \phi_0 + \left(1+\theta_s L^{24}\right)\varepsilon_t \\
\Delta_{24} Y_t &= \phi_0 + \phi_s \Delta_{24} Y_{t-24} + \theta_s \varepsilon_{t-24} + \varepsilon_t \\
Y_t &= \phi_0 + (1+\phi_s) Y_{t-24} - \phi_s Y_{t-48} + \theta_s \varepsilon_{t-24} + \varepsilon_t
\end{aligned}
$$