# The Bootstrap

The Econometrics of Predictability

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# Course Overview

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- Part I: Many Predictions
  - The Bootstrap
  - Technical Trading Rules
  - Formalized Data Snooping: Reality Check and the Test of Superior Predictive Ability
  - False Discovery, Stepwise Testing and the Model Confidence Set
- Part II: Many Predictors
  - Dynamic Factor Models
    - The Kalman Filter
    - Expectations Maximization Algorithm
  - Partial Least Squares and The 3 Pass Regression Filter
  - Regularized Reduced Rank Regression
  - LASSO

# Course Assessment

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- 2 Assignments
  1. Group Work
     - Group of 2
     - 40% of course
     - If odd number of students, 1 group of 3 allowed
     - Empirical
     - Due Friday Week 9, 12:00 at SBS
  2. Individual Work
     - Formal Assignment
     - 60% of course
     - Empirical
     - Due Friday Week 9, 12:00 (Informal)
- Both assignment will make extensive use of MATLAB
- Presentation and content of results counts – code is not important
- Weekly problems to work on will be distributed – a subset of these will compromise the assigned material

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## Definition (The Bootstrap)

The bootstrap is a statistical procedure where data is resampled, and the resampled data is used to estimate quantities of interest.

- Bootstraps come in many forms
  - Structure
    - Parametric
    - Nonparametric
  - Dependence Type
    - IID
    - Wild
    - Block and other for dependent data
- All share common structure of using simulated random numbers in combination with original data to compute quantities of interest
- Applications
  - Confidence Intervals
  - Refinements
  - Bias estimation

# Basic Problem

- Compute standard deviation for an estimator
- For example, in case of mean $\bar{x}$ for i.i.d. data, we know

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

  is usually a reasonable estimator of the standard deviation of the data
- The standard error of the mean is then

$$\mathrm{V}[\bar{x}] = \frac{s^2}{n}$$

  which can be used to form confidence intervals or conduct hypothesis tests (in conjunction with CLT)
- How could you estimate the standard error for the median of $x_1, \ldots x_n$?
- What about inference about a quantile, for example that $5^{\text{th}}$ percentile of $x_1, \ldots x_n$?
- Bootstrap is a computational method to construct standard error estimates of confidence interval for a wide range of estimators.

# IID Bootstrap

- Assume $n$ i.i.d. random (possibly vector valued) variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$
- Estimator of of a parameter of interest $\hat{\theta}$
  - For example, the mean

## Definition (Empirical Distribution Function)

The empirical distribution function assigns probability $1/n$ to each observation value. For a scalar random variable $x_i$, $i = 1, \ldots, n$, the EDF is defined

$$\hat{F}(X) = \frac{1}{n} \sum_{i=1}^{n} I_{[x_i < X]}.$$

- Also known as the empirical CDF
- CDF of $X$ should have information about precision of $\hat{\theta}$, so ECDF might also

# IID Bootstrap for the mean

## Algorithm (IID Bootstrap)

1. *Simulate a set of n i.i.d. uniform random integers* $u_i$, $i = 1, \ldots n$ *from the range* $1, \ldots, n$ *(with replacement)*
2. *Construct a bootstrap sample* $x_b^\star = \{x_{u_1}, x_{u_2}, \ldots x_{u_n}\}$
3. *Compute the mean*

$$\hat{\theta}_b^\star = \frac{1}{n} \sum_{i=1}^{n} x_{b,i}^\star$$

4. *Repeat steps 1–3* $B$ *times*
5. *Estimate the standard of* $\hat{\theta}$ *using*

$$\frac{1}{B} \sum_{i=1}^{B} \left(\theta_b^\star - \hat{\theta}\right)^2$$

# MATLAB Code for IID Bootstrap

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```
n = 100; x = randn(n,1);
% Mean of x
mu = mean(x);
B = 1000;
% Initialize muStar
muStar = zeros(B,1);
% Loop over B bootstraps
for b=1:B
    % Uniform random numbers over 1...n
    u = ceil(n*rand(n,1));
    % x-star sample simulation
    xStar = x(u);
    % Mean of x-star
    muStar(b) = mean(xStar);
end
s2 = 1/(n-1)*sum((x-mu).^2);
stdErr = s2/n
bootstrapStdErr = mean((muStar-mu).^2)
```

# How many bootstrap replications?

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- $B$ is used for the number of bootstrap replications
- Bootstrap theory assumes $B \to \infty$ quickly
- This ensures that the bootstrap distribution is identical to the case where all unique bootstraps were computed
  - There are a lot of unique bootstraps
  - $n^n$ in the i.i.d. case
- Using finite $B$ adds some extra variation since two bootstraps with the same data won't produce identical estimates
- Note: Often useful to set the state of your random number generator so that results are reproducible

```
% A non-negative integer
seed = 26031974
rng(seed)
```

- Should choose $B$ large enough that the *Monte Carlo error* is negligible
- In practice little reason to use less than $1,000$ replications

# Getting the most out of $B$ bootstrap replications

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- Balanced resampling
  - In standard i.i.d. bootstrap, some values will inevitibly appear more than others
  - Balanced resampling ensures that all values appear the same number of times
  - In practice simple to implement

## Algorithm (IID Bootstrap with Balanced Resampling)

1. *Replicate the data so that there are $B$ copies of each $x_i$. The data set should have $Bn$ observations*
2. *Construct a random random permutation of the numbers $1, \ldots, Bn$ as $u_1, \ldots u_{Bn}$*
3. *Construct the bootstrap sample $x_b^\star = \left\{x_{u_{n(b-1)+1}}, x_{u_{n(b-1)+2}}, \ldots x_{u_{n(b-1)+n}}\right\}$*

- This algorithm samples *without replacement* from the replicated dataset of $Bn$ observations
- Each data point will appear exactly $B$ times in the $B$ bootstrap samples

# MATLAB Code for IID Balanced Bootstrap

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```
n = 100; x = randn(n,1);
% Replicate the data
xRepl = repmat(x,B,1);
B = 1000;
% Random permutaiton of 1,...,B*n
u = randperm(n*B);
% Loop over B bootstraps
for b=1:B
    % Uniform random numbers over 1...n
    ind = n*(b-1)+(1:n);
    xb = xRepl(u(ind));
end
```

# Getting the most out of $B$ bootstrap replications

- Antithetic Random Variables
- If samples are *negatively* correlated, variance of statistics can be reduced
  - Basic idea is to order data so that if one sample has too many large values of $x$, then the next will have too many small
  - This can induce negative correlation while not corrupting bootstrap

## Algorithm (IID Bootstrap with Antithetic Resampling)

1. *Order the data so that* $x_1 \leq x_2 \ldots \leq x_n$. *Treat these indices as the original data.*
2. *Simulate a set of* $n$ *i.i.d. uniform random integers* $u_i$, $i = 1, \ldots n$ *from the range* $1, \ldots, n$ *(with replacement)*
3. *Construct the bootstrap sample* $x_b^\star = \{x_{u_1}, x_{u_2}, \ldots x_{u_n}\}$
4. *Construct* $\tilde{u}_i = n - u_i + 1$
5. *Construct the antithetic bootstrap sample* $x_{b+1}^\star = \{x_{\tilde{u}_1}, x_{\tilde{u}_2}, \ldots x_{\tilde{u}_n}\}$
6. *Repeat for* $b = 1, 3, \ldots, B - 1$

- Using antithetic random variables is a general principle applicable to virtually all simulation estimators

# MATLAB Code for IID Bootstrap with Antithetic RV

```
n = 100; x = randn(n,1);
% Mean of x
mu = mean(x);
B = 1000;
% Initialize muStar
muStar = zeros(B,1);
% Sort x
x = sort(x);
% Loop over B bootstraps
for b=1:2:B
    % Uniform random numbers over 1...n
    u = ceil(n*rand(n,1)); xStar = x(u);
    % Mean of x-star
    muStar(b) = mean(xStar);
    % Uniform random numbers over 1...n
    u = n-u+1; xStar = x(u);
    % Mean of x-star
    muStar(b+1) = mean(xStar);
end
corr(muStar(1:2:B),muStar(2:2:B))
```

# Bootstrap Estimation of Bias

- Many statistics have a *finite sample bias*
- This is equivalent to saying that $\hat{\theta} - \theta \approx c/n$ for some $c \neq 0$
  - Many estimators have $c = 0$, for example the sample mean
  - These estimators are unbiased
- Biased estimators usually arise when the estimator is a non-linear function of the data
- Bootstrap can be used to estimate the bias, and the estimate can be used to debias the original estimate
- Recall the definition of bias

## Definition (Bias)

The bias of an estimator is

$$\mathrm{E}\left[\hat{\theta} - \theta\right]$$

# Bootstrap Estimation of Bias

## Algorithm

1. *Estimate the parameter of interest* $\hat{\theta}$
2. *Generate a bootstrap sample* $x_b$ *and estimate the parameter on the bootstrap sample. Denote this estimate as* $\hat{\theta}_b^\star$
3. *Repeat 2 a total of* $B$ *times*
4. *Estimate the bias as*

$$Bias = B^{-1} \sum_{i=1}^{B} \hat{\theta}_b^\star - \hat{\theta}$$

- Example of bootstrap bias adjustment will be given later once more results for time-series have been established

# Bootstrap Estimation of Standard Error

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## Algorithm

1. *Estimate the parameter of interest* $\hat{\theta}$
2. *Generate a bootstrap sample* $x_b$ *and estimate the parameter on the bootstrap sample. Denote this estimate as* $\hat{\theta}_b^\star$
3. *Repeat 2 a total of* $B$ *times*
4. *Estimate the standard error as*

$$\textit{Std. Err} = \sqrt{B^{-1} \sum_{i=1}^{B} \left(\hat{\theta}_b^\star - \hat{\theta}\right)^2}$$

- Other estimators are also common

$$\text{Std. Err} = \sqrt{(B-1)^{-1} \sum_{i=1}^{B} \left(\hat{\theta}_b^\star - \overline{\hat{\theta}_b^\star}\right)^2}$$

- $B$ should be sufficiently large that $B$ or $B-1$ should not matter

# Bootstrap Estimation of Confidence Intervals

- Bootstraps can also be used to construct confidence intervals
- Two methods:
  1. Estimate the standard error of the estimator and use a CLT
  2. Estimate the confidence interval directly using the bootstrap estimators $\left\{\hat{\theta}_b^\star\right\}$
- The first method is simple and have previously been explained
- The second is also very simple, and is known as the *percentile method*

# Percentile Method

## Algorithm (Percentile Method)

*A confidence interval* $\left[C_{\alpha_L}, C_{\alpha_H}\right]$ *with coverage* $\alpha_H - \alpha_L$ *can be constructed:*

1. *Construct a bootstrap sample* $x_b$
2. *Compute the bootstrap estimate* $\hat{\theta}_b^\star$
3. *Repeat steps 1–2*
4. *The confidence interval is constructed using the empirical* $\alpha_L$ *quantile and the empirical* $\alpha_H$ *quantile of* $\left\{\hat{\theta}_b^\star\right\}$

- If the bootstrap estimates are ordered from smallest to largest, and $B\alpha_L$ and $B\alpha_H$ are integers, then the confidence interval is

$$\left[\hat{\theta}^\star_{B\alpha_L}, \hat{\theta}^\star_{B\alpha_H}\right]$$

- This method may not work well in all situations
  - $n$ small
  - Highly asymmetric distribution

# MATLAB Code for Percentile Method

```
n = 100; x = randn(n,1);
% Mean of x
mu = mean(x);
B = 1000;
% Initialize muStar
muStar = zeros(B,1);
% Loop over B bootstraps
for b=1:B
    % Uniform random numbers over 1...n
    u = ceil(n*rand(n,1));
    % x-star sample simulation
    xStar = x(u);
    % Mean of x-star
    muStar(b) = mean(xStar);
end
alphaL = .05;alphaH=.95;
muStar = sort(muStar);
CI = [muStar(alphaL*B) muStar(round(alphaH*B))]
CI - mu
```

# Bootstrap and Regression

- Bootstraps can be used in more complex scenarios
- One simple extension is to regressions
- Using a model, rather than estimating a simple statistic, allows for a richer set of bootstrap options
  - Parametric
  - Non-parametric
- Basic idea, however, remains the same:
  - Simulate random data from the same DGP
  - Now requires data for both the regressor $y$ and the regressand $\mathbf{x}$

# Parametric vs. Non-parametric Bootstrap

- Parametric bootstraps are based on a model
- They exploit the structure of the model to re-sample residuals rather than the actual data
- Suppose

$$y_i = \mathbf{x}_i \boldsymbol{\beta} + \epsilon_i$$

  where $\epsilon_i$ is homoskedastic
- The parametric bootstrap would estimate the model and the residuals as

$$\hat{\epsilon}_i = y_i - \mathbf{x}_i \hat{\boldsymbol{\beta}}$$

- The bootstrap would then construct the re-sampled "data" by sampling $\hat{\epsilon}_i$ separately from $\mathbf{x}_i$
  - In other words, use two separate sets of i.i.d. uniform indices
- Construct $y^{\star}_{b,i} = \mathbf{x}_{u_{1i}} \hat{\boldsymbol{\beta}} + \hat{\epsilon}_{u_{2i}}$
- Compute statistics using these values

# Useful function: bsxfun

- Many examples use bsxfun

```
BSXFUN  Binary Singleton Expansion Function

C = BSXFUN(FUNC,A,B) applies the element-by-element binary
operation specified by the function handle FUNC to arrays A and
B, with singleton expansion enabled. FUNC must be able to accept
as input either two column vectors of the same size, or one
column vector and one scalar, and return as output a column
vector of the same size as the input(s).  FUNC can either be a
function handle for an arbitrary function satisfying the above
conditions or one of the following built-in:
```

- Allows k by n matrix to be added/subtracted from k by 1 vector or 1 by n vector

```
x = randn(1000,10);
mu = mean(x);
err = bsxfun(@minus,x,mu);
```

# MATLAB Code for Parametric Bootstrap of Regression

```
n = 100; x = randn(n,2); e = randn(n,1); y = x*ones(2,1) + e;
% Bhat
Bhat = x\y; ehat = y - x*Bhat;
B = 1000;
% Initialize BStar
BStar = zeros(B,2);
% Loop over B bootstraps
for b=1:B
    % Uniform random numbers over 1...n
    uX = ceil(n*rand(n,1)); uE = ceil(n*rand(n,1));
    % x-star sample simulation
    xStar = x(uX,:); eStar = e(uE);
    yStar = xStar*Bhat + eStar;
    % Mean of x-star
    BStar(b,:) = (xStar\yStar)';
end
Berr=bsxfun(@minus,BStar,Bhat');
bootstrapVCV = Berr'*Berr/B
trueVCV = eye(2)/100
OLSVCV = (e'*e)/n * inv(x'*x)
```

# Non-parametric Bootstrap

- Non-parametric bootstrap is simpler
- It does not use the structure of the model to construct artificial data
- The vector $[y_i, \mathbf{x}_i]$ is instead directly re-sampled
- The parameters are constructed from the pairs

## Algorithm (Non-parametric Bootstrap for i.i.d. Regression Data)

1. *Simulate a set of n i.i.d. uniform random integers* $u_i$, $i = 1, \ldots n$ *from the range* $1, \ldots, n$ *(with replacement)*
2. *Construct the bootstrap sample* $\mathbf{z}_b = \{y_{u_i}, \mathbf{x}_{u_i}\}$
3. *Estimate the bootstrap* $\boldsymbol{\beta}$ *by fitting the model*

$$y_{u_i} = \mathbf{x}_{u_i} \hat{\boldsymbol{\beta}}^{\star}_b + \epsilon^{\star}_{b,i}$$

# MATLAB Code for Nonparametric Bootstrap of Regression

```
n = 100; x = randn(n,2); e = randn(n,1); y = x*ones(2,1) + e;
% Bhat
Bhat = x\y; ehat = y - x*Bhat;
B = 1000;
% Initialize BStar
BStar = zeros(B,2);
% Loop over B bootstraps
for b=1:B
    % Uniform random numbers over 1...n
    u = ceil(n*rand(n,1));
    % x-star sample simulation
    yStar = y(u);
    xStar = x(u,:);
    % Mean of x-star
    BStar(b,:) = (xStar\yStar)';
end
Berr=bsxfun(@minus,BStar,Bhat');
bootstrapVCV = Berr'*Berr/B
trueVCV = eye(2)/100
OLSVCV = (e'*e)/n * inv(x'*x)
```

# Bootstrapping Time-series Data

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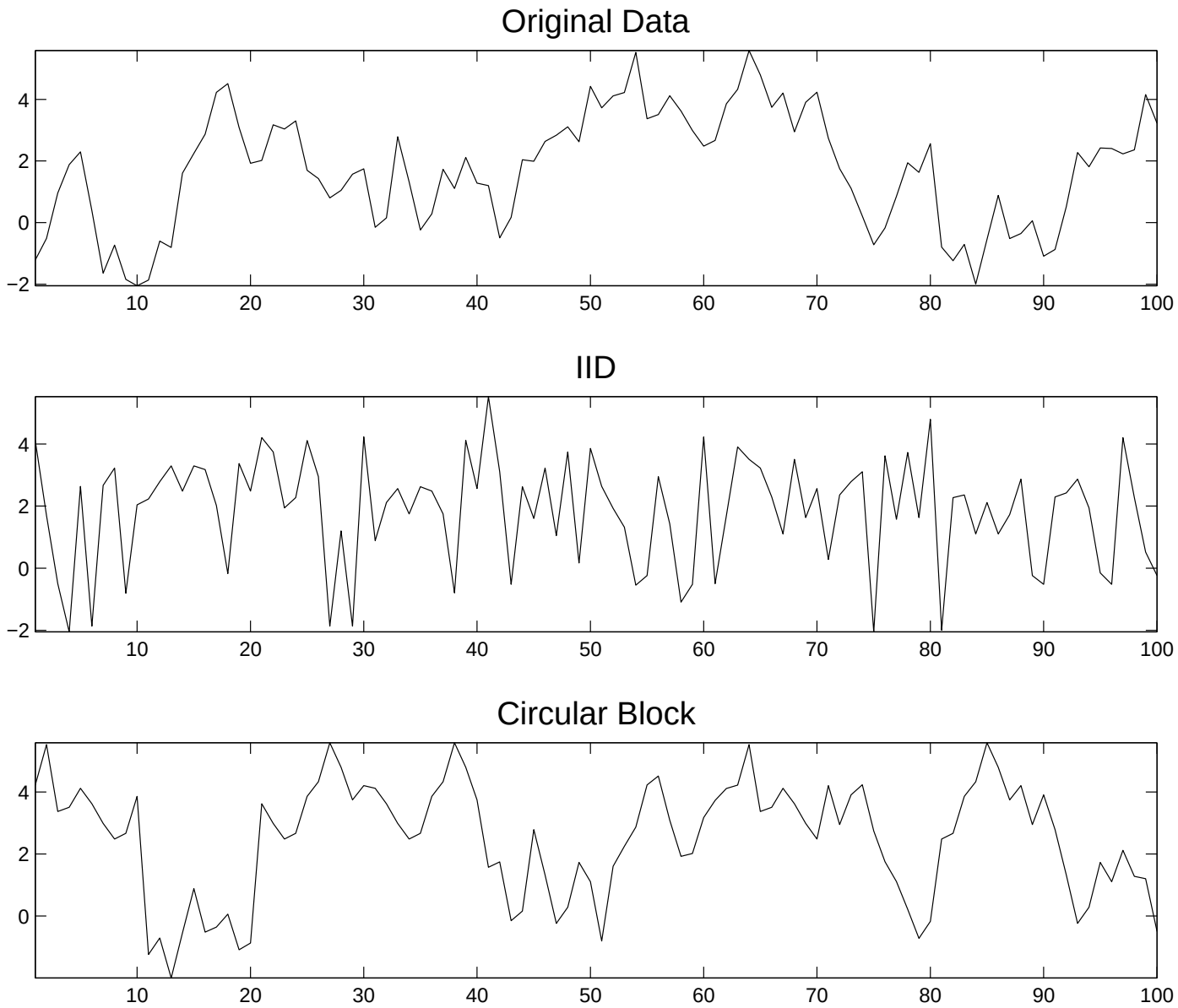
- i.i.d. bootstrap is only appropriate for i.i.d. data
  - Note: Usually OK for data that is not serially correlated
- Two strategies for bootstrapping time-series data
  - Parametric & i.i.d. bootstrap: If the model postulates that the residuals are i.i.d. or at least white noise, then a residual-based i.i.d. bootstrap may be appropriate
    - Examples: AR models, GARCH models using appropriately standardized residuals
  - Nonparametric *block* bootstrap: Weak assumptions, basically that blocks can be sampled so that they (blocks) are approximately i.i.d.
    - Similar to the notion of ergodicity which is related to asymptotic independence
    - Important: Like Newey-West covariance estimator, *block length* must grow with sample size
    - Fundamentally same reason

# The problem with the IID Bootstrap



# MATLAB Code for all time-series applications

```
% Number of time periods
T = 100;
% Random errors
e = randn(T,1);
y = zeros(T,1);
% Y is an AR(1), phi1 = 0.5
y(1) = e(1)*sqrt(1/(1-.5^2));
for t=2:T
    y(t)=0.5*y(t-1)+e(t);
end
% 10,000 replications
B = 10000;
% Initial place for mu-star
muStar = zeros(B,1);
```

# Moving Block Bootstrap

- Samples blocks of $m$ consecutive observations
- Uses blocks which start at indices $1, \dots T - m + 1$

## Algorithm (Moving Block Bootstrap)

1. *Initialize* $i = 1$
2. *Draw a uniform integer* $v_i$ *on* $1, \dots, T - m + 1$
3. *Assign* $u_{(i-1)+j} = v_i + j - 1$ *for* $j = 1, \dots, m$
4. *Increment* $i$ *and repeat 2–3 until* $i \geq \lceil T/m \rceil$
5. *Trim* $u$ *so that only the first* $T$ *remain if* $T/m$ *is not an integer*

# MATLAB Code for Moving Block Bootstrap

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```
% Block size
m = 10;
% Loop over B bootstraps
for b=1:B
    % ceil(T/m) Uniform random numbers over 1...T-m+1
    u = ceil((T-m+1)*rand(ceil(T/m),1));
    u = bsxfun(@plus,u,0:m-1)';
    % Transform to col vector, and remove excess
    u = u(:); u = u(1:T);
    % y-star sample simulation
    yStar = y(u);
    % Mean of y-star
    muStar(b) = mean(yStar);
end
```

# Circular Bootstrap

- Simple extension of MBB which assumes the data live on a circle so that $y_{T+1} = y_1$, $y_{T+2} = y_2$, etc.
- Has better finite sample properties since all data points get sampled with equal probability
- Only step 2 changes in a very small way

## Algorithm (Circular Block Bootstrap)

1. *Initialize* $i = 1$
2. *Draw a uniform integer* $v_i$ *on* $1, \ldots, T$
3. *Assign* $u_{(i-1)+j} = v_i + j - 1$ *for* $j = 1, \ldots, m$
4. *Increment* $i$ *and repeat 2–3 until* $i \geq \lceil T/m \rceil$
5. *Trim* $u$ *so that only the first* $T$ *remain if* $T/m$ *is not an integer*

# MATLAB Code for Circular Block Bootstrap

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```
% Block size
m = 10;
% Loop over B bootstraps
yRepl = [y;y];
for b=1:B
    % ceil(T/m) Uniform random numbers over 1...T-m+1
    u = ceil(T*rand(ceil(T/m),1));
    u = bsxfun(@plus,u,0:m-1)';
    % Transform to col vector, and remove excess
    u = u(:); u = u(1:T);
    % y-star sample simulation
    yStar = yRepl(u);
    % Mean of y-star
    muStar(b) = mean(yStar);
end
```

# Stationary Bootstrap

- Differs form MBB and CBB in that the block size is no longer fixed
- Chooses an average block size of $m$ rather than an exact block size
- Randomness in block size is worse when $m$ is known, but helps if $m$ may be suboptimal
- Block size is *exponentially distributed* with mean $m$

## Algorithm (Stationary Bootstrap)

1. Draw $u_1$ uniform on $1, \ldots, T$
2. For $i = 2, \ldots, t$
   a. Draw a uniform $v$ on $(0, 1)$
   b. If $v \geq 1/m$ $u_i = u_{i-1} + 1$
      i. If $u_i > T$, $u_i = u_i - T$
   c. If $v < 1/m$, draw $u_i$ uniform on $1, \ldots, T$

# MATLAB Code for Stationary Bootstrap

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```
% Average block size
m = 10;
% Loop over B bootstraps
yRepl = [y;y];
u = zeros(T,1);
for b=1:B
    u(1) = ceil(T*rand);
    for t=2:T
        if rand<1/m
            u(t) = ceil(T*rand);
        else
            u(t) = u(t-1) + 1;
        end
    end
    % y-star sample simulation
    yStar = yRepl(u);
    % Mean of y-star
    muStar(b) = mean(yStar);
end
```

# Comparing the Three TS Bootstraps

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- MBB was the first
- CBB has simpler theoretical properties and usually requires fewer corrections to address "end effects"
- SB is theoretically worse than MBB and CBB, but is the most common choice in time-series econometrics
  - Theoretical optimality assumes that the the "optimal" block size is used
- Popularity of SB stems from difficulty in determining optimal $m$
  - More on this in a minute
- Random block size brings some robustness at the cost of extra variability

# Bootstrapping Stationary AR(P)

- The stationary AR(P) model can be parametrically bootstraps
- Assume

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_P y_{t-P} + \epsilon_t$$

- Usual assumptions, including stationarity
- Can use a parametric bootstrap by estimating the residuals

$$\hat{\epsilon}_t = y_t - \hat{\phi}_1 y_{t-1} + \ldots + \hat{\phi}_P y_{t-P}$$

## Algorithm (Stationary Autoregressive Bootstrap)

1. *Estimate the AR(P) and the residuals for* $t = P+1, \ldots, T$
2. *Recenter the residuals so that they have mean 0*

$$\tilde{\epsilon}_t = \hat{\epsilon}_t - \bar{\hat{\epsilon}}$$

3. *Draw u uniform from* $1, \ldots, T-P+1$ *and set* $y_1^\star = y_u$, $y_2^\star = y_{u+1}, \ldots, y_P^\star = y_{u+P+1}$
4. *Recursively simulate* $y_{P+1}^\star \ldots y_T^\star$ *using* $\tilde{\epsilon}$ *drawn using an i.i.d. bootstrap*

# MATLAB Code for Stationary AR Bootstrap

```
phi = y(1:T-1)\y(2:T);
ehat = y(2:T)-phi*y(1:T-1);
etilde = ehat-mean(ehat);
yStar = zeros(T,1);
for i=1:B
    % Initialize to one of the original values
    yStar(1) = y(ceil(T*rand));
    % Indices for errors
    u = ceil((T-1)*rand(T,1));
    % Recursion to simulate AR
    for t=2:T
       yStar(t) = phi*yStar(t-1) + ehat(u(t));
    end
end
```

# Data-based Block Length Selection

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- Block size selection is crucial for good performance of block bootstraps
- Small block sizes are too close to i.i.d. while large block sizes are overly noisy
- Politis and White (2004) provide a data dependent lag length selection procedure
  - See also Patton, Politis, and White (2007) correction
- Code is available by searching the internet for "opt_block_length_REV_dec07"

# How Data-based Block Length Selection Works

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- Politis and White (2004) show for stationay bootstrap

$$B_{opt,SB} = \left(\frac{2G^2}{D_{SB}}\right) N^{1/3}$$

  - $G = \sum_{k=-\infty}^{\infty} |k| \gamma_k$ where $\gamma_k$ is the autocovariance
  - $D_{SB} = 2g(0)^2$ where $g(w) = \sum_{s=-\infty}^{\infty} \gamma_s \cos(ws)$ is the spectral density function
- Need to estimate $\hat{G}$ and $\hat{D}_{SB}$ to estimate $\hat{B}_{opt,SB}$
  - $\hat{G} = \sum_{k=-M}^{M} \lambda\left(k/M\right) |k| \hat{\gamma}_k$,

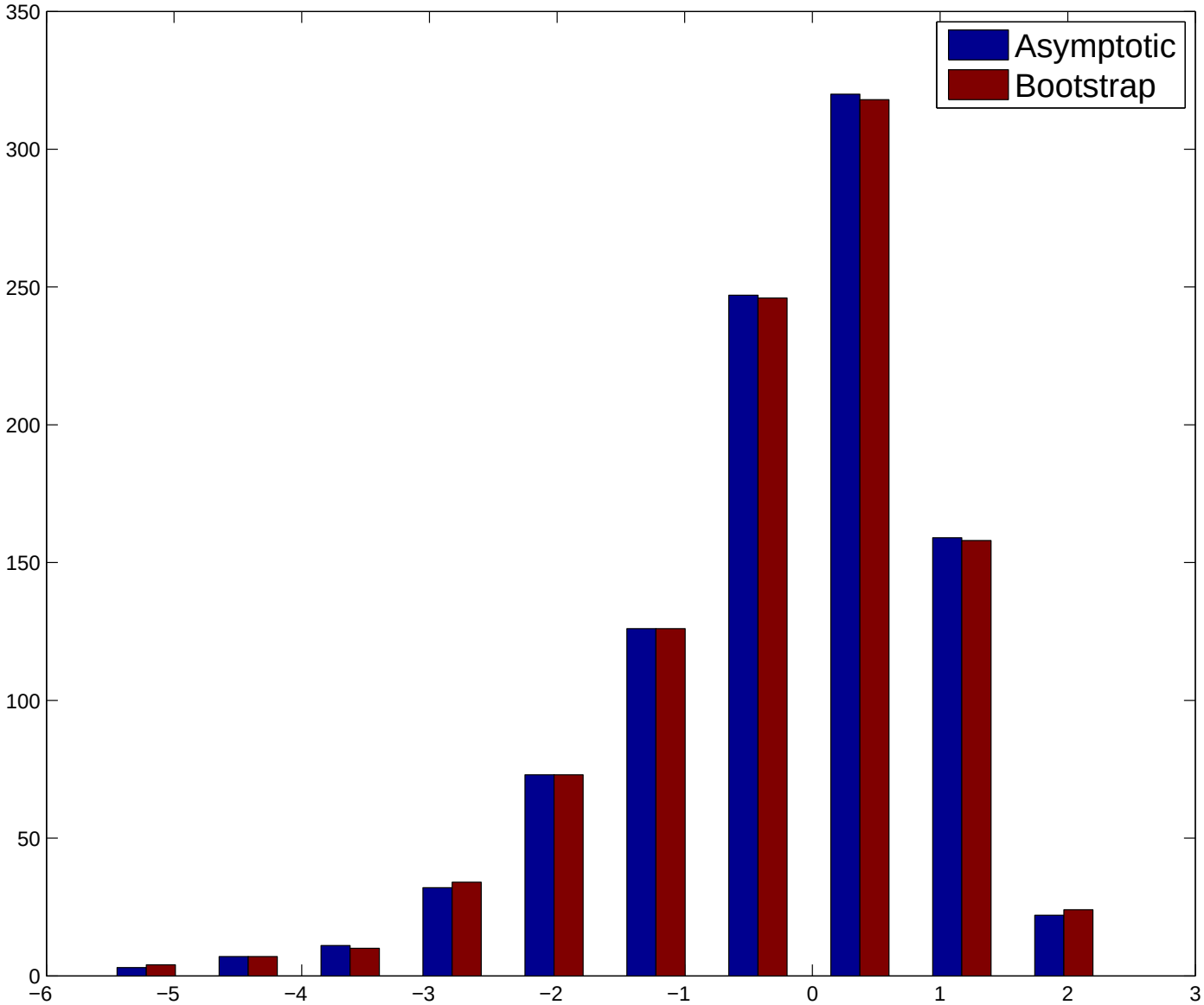
$$\lambda(s) = \begin{cases} 1 & \text{if } |s| \in [0, 1/2] \\ 2\left(1 - |s|\right) & \text{if } |s| \in [1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

  - $\hat{D}_{SB} = 2\hat{g}(0), \hat{g}(w) = \sum_{k=-M}^{M} \lambda\left(k/M\right) \hat{\gamma}_k \cos\left(wk\right)$
  - $M$ is set to $2\hat{m}$
  - $\hat{m}$ is the smallest integer where if $\hat{\rho}_j > 2\sqrt{\log T/T}, j = m+1, \ldots, K_T$ where $K_T = 2\max\left(5, \sqrt{\log_{10}(T)}\right)$

# Example 1: Mean Estimation for Log Normal

- $y_i \overset{\text{i.i.d.}}{\sim} LN(0, 1)$
- $n = 100$
- $B = 1000$ using i.i.d. bootstrap
- This is a check that the bootstrap works
- Also shows that bootstrap will not work miracles
- Performance of bootstrap is virtually identical to that of asymptotic theory
  - Gains to bootstrap are more difficult to achieve
  - Most useful property is in estimating standard error in hard to compute cases

# Example 1: Mean Estimation for Log-Normal



# Example 2: Bias in AR(1)

- Assume $y_t = \phi y_{t-1} + \epsilon_t$ where $\epsilon_t \overset{\text{i.i.d.}}{\sim} N(0, 1)$
- $\phi = 0.9$, $T = 50$
- Use parametric bootstrap
- Estimate bias using the different between bootstrap estimates and the actual estimate

| | Direct | Debiased |
|---|---|---|
| $\hat{\phi}$ | 0.8711 | 0.8810 |
| Var | 0.0052 | 0.0044 |

- Reduced the bias by about 1/3
- Reduced variance (rare)