

Size and Power of a Test of the Mean with Normal Data

Suppose n i.i.d. normal random variables have unknown mean μ but known variance σ^2 and so the sample mean, $\bar{y} = n^{-1} \sum_{i=1}^n y_i$, is then distributed $N(\mu, \sigma^2/N)$. When testing a null that $H_0 : \mu = \mu_0$ against an alternative $H_1 : \mu \neq \mu_0$, the size of the test is the probability that the null is rejected when it is true. Since the distribution under the null is $N(\mu_0, \sigma^2/N)$ and the size can be set to α by selecting points where $\Pr(\hat{\mu} \in (\underline{C}_\alpha, \bar{C}_\alpha) | \mu = \mu_0) = 1 - \alpha$. Since the distribution is normal, one natural choice is to select the points symmetrically so that $\underline{C}_\alpha = \mu_0 + \frac{\sigma}{\sqrt{N}} \Phi^{-1}(\alpha/2)$ and $\bar{C}_\alpha = \mu_0 + \frac{\sigma}{\sqrt{N}} \Phi^{-1}(1 - \alpha/2)$ where $\Phi(\cdot)$ is the CDF of a standard normal.

The power of the test is defined as the probability the null is rejected when the alternative is true. This probability will depend on the population mean, μ_1 , the sample size, the test size and mean specified by the null hypothesis. When testing using an α -sized test, rejection will occur when $\hat{\mu} < \mu_0 + \frac{\sigma}{\sqrt{N}} \Phi^{-1}(\alpha/2)$ or $\hat{\mu} > \mu_0 + \frac{\sigma}{\sqrt{N}} \Phi^{-1}(1 - \alpha/2)$. Since under the alternative $\hat{\mu}$ is $N(\mu_1, \sigma^2)$, these probabilities will be

$$\Phi\left(\frac{\mu_0 + \frac{\sigma}{\sqrt{N}} \Phi^{-1}(\alpha/2) - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right) = \Phi\left(\frac{\underline{C}_\alpha - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right)$$

and

$$1 - \Phi\left(\frac{\mu_0 + \frac{\sigma}{\sqrt{N}} \Phi^{-1}(1 - \alpha/2) - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right) = 1 - \Phi\left(\frac{\bar{C}_\alpha - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right).$$

The total probability that the null is rejected is known as the power function,

$$\text{Power}(\mu_0, \mu_1, \sigma, \alpha, N) = \Phi\left(\frac{\underline{C}_\alpha - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right) + 1 - \Phi\left(\frac{\bar{C}_\alpha - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right).$$

Bibliography

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